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Economic Scenario Generators: a risk management tool for insurance

Pierre-Edouard Arrouy *
Alexandre Boumezoued **
Bernard Lapeyre * * *
Sophian Mehalla †

* Milliman, 14 Avenue de la Grande Armée, 75017, Paris, France
E-mail address: pierre-edouard.arrouy@milliman.com
** Milliman, 14 Avenue de la Grande Armée, 75017, Paris, France
E-mail address: alexandre.boumezoued@milliman.com
* * * École des Ponts ParisTech, 6-8 avenue Blaise-Pascal 77455 Marne-la-Vallée, France
E-mail address: bernard.lapeyre@enpc.fr
† Milliman, 14 Avenue de la Grande Armée, 75017, Paris, France
E-mail address: sophian.mehalla@milliman.com.

Abstract

We present a risk management tool, named Economic Scenario Generator (ESG), used by insurance companies for simulating the global state of one or several economies described by key financial risk drivers. This tool is of particular use within the Solvency II framework, since insurance companies are required to value their balance-sheet from a *market-consistent* viewpoint. However, there is no observable price of insurance contracts hence the necessity of relying on ESGs to perform Monte Carlo simulations useful for valuation. As such, the calibration of Risk-Neutral models underlying this valuation is of particular interest as there is a strong requirement to match observable market prices. Furthermore, for a variety of applications, the insurance company has to value its balance-sheet over a set of different economic conditions, leading to the need of intensive re-calibrations of such models. In this paper, we first provide an overview of the key requirements from Solvency II and their practical implications for insurance valuation. We then describe the different use cases of ESGs. A particular attention is paid to Risk-Neutral interest rates models, specifically the Libor Market Model with a stochastic volatility. We discuss the complexity of its calibration and describe fast calibration methods based on approximations and expansions of the probability density function. Comparisons with more common method highlight the reduction in calibration time.

1. Introduction

The insurance activity is basically the management of a large class of risks: natural, biometric, human behavior, financial, etc. The contracts issued by the insurers aim at transferring the risk from the policyholder to the insurance company in exchange of the payment of a risk premium. The amount of the risk premium to be paid depends on a number of parameters: the policyholder (age, health, etc.), the risks that are involved in the contract and in the strategy of the insurance company to back this contract. The guarantees and options embedded in the contracts are such that the pricing of the contracts is a complex task. As in bank industry, stochastic simulations of risks drivers constitute a common method to do so.

Regarding the financial risk in particular, the type of models that are used are similar to those used in bank industry. This is a quite recent choice mainly motivated by the legislation in Solvency II. It originated after the 2008 financial crisis and it has been thought to consolidate the insurance sector. Among models used to simulate financial risk drivers, those dedicated to interet-rates have reached a significant degree of complexity. Notably, the quite common model named

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Displaced Diffusion with Stochastic Volatility LIBOR Market Model (DDSVLMM) has focused the attention of practitioners. In particular, the significant time required for its calibration was a problem in practice.

The purpose of this paper is to explain why and how stochastic simulations are central for insurance undertakings to pursue their activity. In particular, simulations of financial risk drivers are realized by Economic Scenario Generators that are tools designed especially for insurance needs. We present these tools and some regulatory elements in Section 2. In Section 3, we present some recent works related to the speeding of the calibration of DDSVLMM.

2. **Economic Scenario Generators**

Insurance is an old activity that basically consists in managing the different kinds of risks the economic agents are exposed to. Insurance policies are designed to protect either individuals or estates and are at the centre of modern economies. The high technicality of economies of more economically developed countries requires the insurers to be able to face a large variety of risks. In particular, those economies are mainly driven by macroeconomic aggregates and financial quantities: the financial risk arises as a key element for activity of companies. Indeed, the backing of life-insurance contracts is composed of financial derivatives explaining why insurers are exposed to market risk.

For risk management, strategic guidance, regulatory compliance, sensitivities computations or valuations of policies, the *Economic Scenario Generators* (ESGs) recently emerged as a must-have tool for insurers. Following [23], an ESG can be defined as "*a computer-based model of an economic environment that is used to produce simulations of the joint behaviour of financial market values and economic variables.*" An ESG comprises several models, dependent on one another, each one being dedicated to the modelling of economic quantities that reflect different risks. Current ESGs can account for interest rates risk, equity risk, credit risk, real estate risk and foreign exchange risk. Note that the very nature of insurance policies embedding optional guarantees of relative long life enlightens why movements in interest rates curve have predominant impact on the whole activity of insurers. Moreover, a major part of the financial instruments used for backing life-insurance contracts corresponds to (Sovereign or Corporate), possibly Zero-Coupon and other interest rates derivatives. Before discussing the design of an ESG in Section 2.4, we provide in Sections 2.1, 2.2 and 2.3 a non-exhaustive list of the main uses of ESGs in insurance.

2.1. **Technical provisions**

Technical provisions (TPs) represent the economic value of the commitments of the insurer written in the contracts that have been issued by the company. Due to the variety of risks faced by insurers and their imbrications, this quantity is hard to calculate. Risks associated to contracts that may be perfectly replicated by a portfolio composed of financial instruments could be valued using the value of this portfolio and in that case, TPs equal the market value of the replicating portfolio. However, most of the insurance contracts can not be perfectly replicated and the assessment of the associated economic value of the liabilities is a complex task therefore a number of approximations are necessary to be able to obtain a value. To do so, simulations of the risk drivers are employed to mimic the behavior of the assets and liabilities of the insurance company.

Such liabilities are valued using a *Best Estimate* (BE) approach (see the dedicated paragraph below). On top of that, a *Risk Margin* (RM) is added to the BE as a prudential stock accounting for the non-replicability of some insurance risks:

$$\text{Technical Provisions} = \text{Best Estimate} + \text{Risk Margin}.$$

The Best Estimate: it is defined in [2] as "*the probability-weighted average of future cash-flows, taking into account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure. The calculation of the best estimate shall be based upon up-to-date and credible information and realistic assumptions and be performed using adequate, applicable and relevant actuarial and statistical methods*". Mentioned cash-flows come from all kinds of risks faced by insurers. They can be split in two categories: financial risks that can be replicated and non-financial ones that are non-hedgeable. In formulas, the BE (also sometimes referred to as BEL standing for *Best Estimate of Liabilities*) expresses as

$$\mathrm{BE} = \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\sum_{n\geq 1} D(0,n)\mathrm{CF}_n\right] \tag{2.1}$$

where $(D(0,n))_{n\geq 1}$ are discounting factors associated to risk-free term structure, $\mathbb{Q}$ is the standard Risk-Neutral probability measure associated to financial risks (usually chosen as being the probability measure whose *numéraire* is the value of the "money market account"), $\mathbb{P}$ is the historical probability measure used for other risks (behaviour of policyholders, mortality, longevity, etc.) and CF_n is the cash-flow delivered at time $n \geq 1$ for the n-th period.

In practice, the BE is estimated through a Monte-Carlo approach using simulations of future cash-flows:

$$\mathrm{BE} = \lim_{L\to\infty} \frac{1}{L}\sum_{l=1}^{L}\sum_{n=1}^{N} D^{(l)}(0,n)\mathrm{CF}_n^{(l)}.$$

For a given simulation l, the sequence $(\mathrm{CF}_n^{(l)})_{1\leq n\leq N}$ comprises cash-in and out-flows so that they can be decomposed as $\mathrm{CF}_n^{(l)} = \mathrm{CF}_n^{(l),\mathrm{out}} - \mathrm{CF}_n^{(l),\mathrm{in}}$ at any time $n \geq 1$. Practitioners use *Assets Liability Management* (ALM) models to compute sequences of cash-flows: those are complex models that take as inputs *simulated paths of the state of the world* and appreciate the interactions between assets and liabilities of the company through the optional guarantees carried out in the insurance policies issued by the company. The paths on which the computations of cash-flows are based are either built from historical data (mortality or longevity tables) or generated by mathematical models gathered in ESGs. Among them, those dedicated to market risk are coming from bank industry: they are calibrated to market data so that the BE depends on late economic condition. Note also that the computation of the BE allows to integrate national politics scheme as ALM models include operative accounting rules in each country.

The interest rates play a particular role in the computation of Best Estimate. It can be directly seen in the definition (2.1) as the sequence of discount factors only depends on interest rates. The joint distribution of interest rates and other risk factors that intervene in the computation of cash-flows is thus key to compute the BE. Moreover, among the risk factors themselves the interest rates are prominent as noted in [11] and [3]: 72.1% of the assets portfolio correspond to bonds (31.6% sovereign and 40.5% corporate) for the representative portfolio of the Euro zone and 72.3% (29.1% sovereign and 43.2% corporate) in France[1].

Risk Margin: "*[it] shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations.*" following [2]. Alternatively, and following [1] or [15], "*risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof*". The RM adds up to the BE to take into account the non-replicability of most of insurer's liabilities. The act states that "*(...) the risk margin shall be calculated by determining the cost of providing an amount of*

[1] Data can be found following `www.eiopa.europa.eu/content/eiopa-updates-representative-portfolios-calculate-volatility-adjustments-solvency-ii-risk_en`.

eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof." Alternatively, it can be interpreted as the amount the shareholders will have to invest in the company during the years to come to allow the company to pursue its activity in respect of the legislation.

In formulas,

$$\mathrm{RM} = r_{CoC} \sum_{t \geq 1} P(0,t)\mathrm{SCR}_t \tag{2.2}$$

where r_{CoC} is the Cost-of-Capital rate, SCR_t is the Solvency Capital Requirement at time t whose definition will be detailed below and $P(0,t)$ is the Zero-Coupon bond that is the spot value of one unit of currency delivered at time t and valued using risk-free rate curve. Note that computation of SCR_t as described in more details in the following paragraph is only permitted for $t = 1$ since for $t > 1$, the exhaustive process would be too heavy computationally speaking. A number of approximations are then necessary. The rate r_{CoC} stands for the cost a company would endure for holding an eligible amount of own funds: it is determined by EIOPA (the European regulator of the insurance sector) and currently set to 6% per year in the statutes.

Example 2.1. Let us draw from an example found in [12] of a Euro contract with a minimum guaranteed rate and a profit sharing clause.

Let us denote by r_g the continuously compounded minimum guaranteed rate, by r_{share} the profit sharing rate, by r_{fees} the rate associated to fees, r_d is the effectively delivered rate to the policyholder, $(r_t)_{t \geq 0}$ is the risk-free short rate and by τ_{red} the redemption date. To ensure the delivery of coupons to the policyholder, the insurer invests policyholder's deposit in a financial asset whose time-t value is denoted by $A(t)$ associated with a return rate $r_{ret}(t) = \ln\left(\frac{A(t)}{A(t-1)}\right)$. The successive rates of return are assumed to be independent. The delivered rate is the maximum rate between the guaranteed rate and the net return rate of the asset so that the discounted payoff of the considered product writes:

$$\text{Payoff} = \sum_{t=0}^{T-1} e^{-\int_0^t r_s \mathrm{d}s} A(0)\Big(1 + \max\left(e^{r_g} - 1, r_{share}(e^{r_{ret}(t)} - 1) - (e^{r_{fees}} - 1)\right)\Big).$$

This payoff depends thus on financial risk drivers that are involved in this contract (namely, the equity and the risk-free rates).

Simulations of those risks drivers allows to simulate the behaviour of this payoff along each of the generated paths and deduce the sequence of cash-flows delivered by this contract in a number of generated states of the world.

2.2. **Regulatory computations**

An important aspect that motivates the use of mathematical based procedures in insurance is the legislation. In particular, the Solvency II legislation that came into effect January 1^{st}, 2016 was initiated after the 2008 financial crisis. It requires the insurer to hold a certain quantity of cash during the lifetime of the insurer's commitments to avoid lack of liquidity that could lead a company to not be able to fulfil its commitments. This quantity is named the *Solvency Capital Requirement* (SCR). It is part of the available cash of the company. The whole amount of available cash forms what is named the *Own Funds* (OF) of the company and is classified according to the degree of availability of the cash. The part of Own Funds that is the "more available" (roughly speaking) is denoted NAV (Net Asset Value) and is used to compute the solvency ratio NAV/SCR. It is an economic indicator useful to deem the solvency of the undertaking: when it is greater than one, the company is considered as being solvable at one year, i.e. being able to fulfil its commitments in the year to come in 99.5% of the states of the world.

Practically, a company endures an economic ruin when the net value of its assets becomes non-positive. The SCR is defined as being the minimum amount the insurance must hold at the date of evaluation so that the probability for the company to endure an economic ruin during the next year is smaller than a 0.5% threshold:

$$\mathrm{SCR} := \inf\{x \in \mathbb{R} : \mathbb{P}(\mathrm{NAV}_1 \leq 0 |\, \mathrm{NAV}_0 = x) \leq 0.005\}, \tag{2.3}$$

where NAV_0 is the value of the NAV at evaluation date and NAV_1 is the random variable representing the net assets value in one year. In practice, the SCR is generally computed as a quantile on the one-year loss distribution through a Value-at-Risk (VaR). The one-year loss valued at time $t = 0$ is the random quantity defined by $L := \mathrm{NAV}_0 - D(0,1)\,\mathrm{NAV}_1$. The SCR may alternatively be defined as the 99.5%-quantile of L:

$$\begin{aligned}
\mathrm{SCR} &= \inf_{x\in\mathbb{R}} \Big\{ \mathbb{P}(L \leq x) \geq 0.995 \Big\} \\
&= \inf_{x\in\mathbb{R}} \Big\{ \mathbb{P}(D(0,1)\,\mathrm{NAV}_1 + (x - \mathrm{NAV}_0) \leq 0) \leq 0.005 \Big\} \\
&= \mathrm{NAV}_0 + \inf_{x\in\mathbb{R}} \Big\{ \mathbb{P}(D(0,1)\,\mathrm{NAV}_1 + x \leq 0) \leq 0.005 \Big\} \\
&= \mathrm{NAV}_0 + \mathrm{VaR}_{0.5\%}(D(0,1)\,\mathrm{NAV}_1\,) \\
&= \mathrm{NAV}_0 - q_{0.5\%}(D(0,1)\,\mathrm{NAV}_1\,),
\end{aligned} \tag{2.4}$$

where the VaR associated to the random variable X at quantile $\lambda \in [0,1]$ is defined as $\mathrm{VaR}_\lambda(X) = \inf_{m\in\mathbb{R}} \Big\{ \mathbb{P}(X + m \leq 0) \leq \lambda \Big\}$ and is linked to the quantile function q *via* the following relationship $\mathrm{VaR}_\lambda(X) = -\sup_{m'\in\mathbb{R}} \Big\{ \mathbb{P}(X < m') \leq \lambda \Big\} =: -q_\lambda(X)$.

In practice, the determination of the distribution of NAV_1 is delicate: the commitments associated with all the contracts issued by the company should be valued. We have seen in Example 2.1 - which is a relatively simple insurance contract - that the economic valuation of insurance policies is complex for a single contract. Thus, at the level of the whole company, some proxies are necessary to aggregate the value of all commitments. Under Solvency II, insurers can choose either to apply the *Standard Formula* or to develop an *Internal Modelling* approach to determine the distribution of the one year loss and compute the SCR: the Standard Formula is a formula provided by the regulator (EIOPA for Euro zone) that decomposes losses according to each kind of risk the company is facing before aggregating them assuming a Gaussian copula type dependency; the Internal modelling approach is a procedure developed by the company itself to reflect the particularities of its portfolio. It is a long procedure, that has to be approved by the regulator. This approach is intended to large companies as it requires significant human resources.

Standard Formula

The computation of the SCR using the standard formula is built following the bottom-up aggregating approach described in Figure 2.1. Elementary risks (interest rates, mortality, equity, etc.) are gathered in different modulus: market, health, life, non-life, default. The modulus “Intangible” accounts for the risks associated to intangible assets (comprising intellectual properties, licenses, etc.).

For each elementary risk (interest rates, equity, natural disasters, etc.), an *economic capital* is computed: it represents the sensitivity of the balance sheet of the company with respect to a marginal variation in the risk factor associated to the considered elementary risk. Note that as defined in Solvency II, those marginal variations are coherent with historical shocks of each risk factor occurring once every 200 years (99.5% of probability of occurrence under historical measure probability). For instance, the economic capital associated to interest rates is a valuation of the risk resulting from a sudden movement in the yield curve. Economic capital is defined as being the difference between the central value of the NAV (obtained using current state of risk

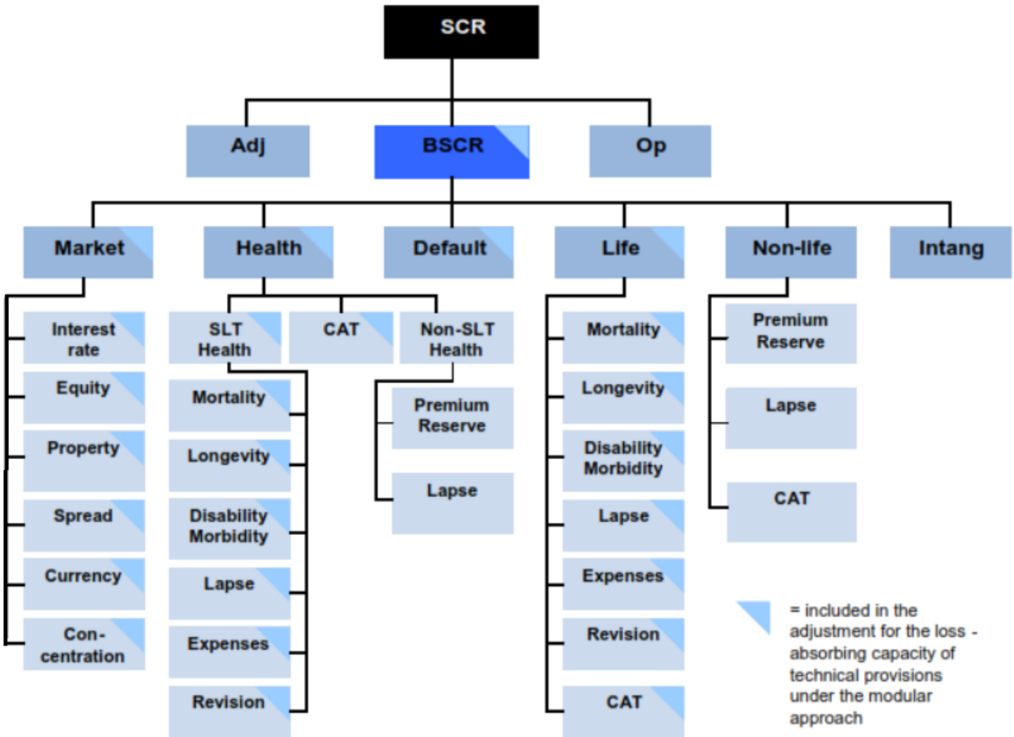


FIGURE 2.1. Structure of the Standard Formula in Solvency II.

factors) and the shocked value of the NAV - obtained when shocking the associated risk factor:

$$\mathrm{EC}_x = \mathrm{NAV}_x^{\mathrm{Central}} - \mathrm{NAV}_x^{\mathrm{Shocked}}$$

for $x \in \{\text{interest rates, equity, mortality}, \dots\}$. Sub-SCRs associated to each modulus are defined by a first aggregation formula

$$\mathrm{SCR}_m = \sqrt{\sum_{(i,j)\in R_m^2} \rho_{i,j}^m \mathrm{EC}_i \mathrm{EC}_j},$$

where R_m is the set of elementary risks in the modulus m and $\rho_{i,j}^m$ is the correlation between elementary risks within modulus m. The correlation matrix $(\rho_{i,j}^m)_{(i,j)\in R_m^2}$ is given by the regulator. Secondly, the aggregation of the sub-SCRs is done in a similar fashion to get the so-called Base SCR (BSCR):

$$\mathrm{BSCR} = \sqrt{\sum_{(m,n)\in M^2} \rho_{m,n}^M \mathrm{SCR}_m \mathrm{SCR}_n},$$

where M is the set of all risks modulus and $(\rho_{m,n}^M)_{(m,n)\in M^2}$ is the correlation matrix between risks modulus, also published by the regulator. The BSCR defined this way does not take into account the operational risks (human errors, system breakdowns, etc.) nor the ability for the insurer to absorb a part of the loss by differing some taxes or by reducing the technical provisions. The operational SCR, denoted by SCR_{op}, and an adjustment are added to the BSCR so that both phenomena are actually integrated in the final SCR:

$$\mathrm{SCR} = \mathrm{BSCR} + \mathrm{SCR}_{op} + \mathrm{Adj}. \tag{2.5}$$

Internal Model

Before going further we give a few insights on the *Internal Model* approach. The previous standard formula is prescribed by the regulator, and thus does not take into account the specificities of the insurers portfolios. To compute their solvency requirements, undertakings can alternatively establish their own methodology. Each step of the proposed methodology should be submitted and justified to the regulatory authority, which makes its validation a very long process and requires important human and operational resources.

The so-called 'nested simulations' approach that underlies Internal Models (although not widespread in practice) is the following (see [10]): under Real-World measure, projections of risk drivers are made over 1 year. At the end of it, calibrations of Risk-Neutral models are performed on those projected economic environments. Simulations under Risk-Neutral measure can then be achieved over the lifetime of the liabilities of the company (several decades). The obtained Risk-Neutral paths are used to derive the distribution of the NAV and compute the SCR as previously stated.

Other forms of Internal Models can be found, that rely on the calibration of a response function based on a sample of points underlying the full distribution, see [21]. In all cases, intensive recalibrations of Risk-Neutral models are generally required, as well as practical alternatives based on simplifications.

2.3. Assets and Liabilities Management (ALM)

ALM modelling consists in the design of models reflecting the interactions between assets and liabilities composing the balance sheet of a company. It should integrate the regulatory rules that apply to the assets and liabilities management but also the particularity of the portfolio of the company. ALM modelling has several goals.

First, such models are used in a forward-looking approach in order to be able to deliver promises as insurers can adapt their investments strategy based on the outputs of those models. Second, they allow to generate the sequence of delivered cash-flows generated by the contracts issued by the company which is necessary in the Best-Estimate computation (see Definition 2.1).

An ALM model is specific to the insurer who designed it as they heed the specific management rules of the company and its portfolio. In the literature, a quite simple ALM model is proposed in [21] and allows to get some intuition about the management rules. More recently, a more elaborated model has been set in [7]. Some issues raised regarding ALM modelling are discussed in [6]. Simplified ALM models are employed by practitioners to realize some specific impact studies.

Simplified ALM model. We provide below some illustrations coming from a simplified insurance cash-flows model. The latter forecasts the behaviour of the company's portfolio with cash in and out flows generated by the issued contracts. The asset part is composed of some available cash, risk-free bonds (issued at par), an equity index and a real estate index. On liability side, and for simplicity, we consider only saving euro contracts with buypack option (as introduced in Example 2.1 with a buypack clause), characterized by a minimum guaranteed rate, a profit sharing rate, the age of the contract, the age of the policyholder and the initial invested premium. The liabilities associated to such kind of contracts are projected in the ALM model thanks to simulated economic paths along with a buypack rule and a mortality table. We assume that the company does not issue new insurance contracts beyond the starting date (the date at which the valuation is realized). Finally, let us mention that at each date, the portfolio is updated to respect a prescribed asset allocation with possible purchases of (risk-free) bonds or equity linked products. Below, we provide in Figure 2.2 the projection over time of the distribution of sequence of cash-flows involved in BE computation (cf. Equation 2.1). In this

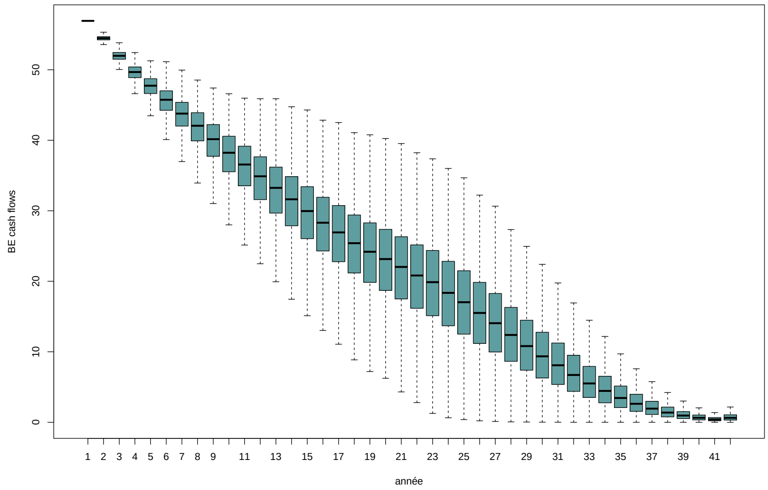


Figure 2.2. Distributions of BE cash-flows through time. Projection is made on the basis of 3000 interest rates, equity and real estate simulations realized thanks to an ESG calibrated on 12/31/2021 market data.

particular example, we see that as time goes, insured portfolio decreases following a number of typical events, such as deaths of policyholder or lapses. The optionality embedded in the issued contracts is anticipated to be maximal between the 17^{th} and 25^{th} years.

2.4. **Designing ESGs**

The design and the content of ESGs depend on the final purpose of the generated paths by ESG. We distinguish three main types of designs for ESGs depending on usage: (i) Risk-Neutral ESGs designed for economic valuation and simulation under probability measure extracted from financial markets; (ii) Real-World ESGs whose purpose is to simulate paths of risks drivers in keeping with historical observations and (iii) ESGs designed for regulatory compliance.

Risk-Neutral ESG. In most of the previously mentioned computations, replication of market data (prices of derivatives or volatilities) is required in order to ensure the consistency with current economic conditions. The translation of this notion for the regulatory requirements is named the *market consistency* (discussed in [24] or [13] and references therein). Furthermore, an important feature of insurance policies is that they embed optional guarantees. Consequently, the simulated paths obtained from ESGs has to be *stochastic* to take into account the value of this optionality coming from the variety of possible behaviours of policyholders. These two points mainly explain why, historically, insurers have been led to consider models coming from bank industry to perform the aforementioned calculations. Even though they were not designed for long-term projections and were mainly motivated by establishment of hedging strategy associated with high frequency[2] of portfolio rebalancing, they offer the ability of stochastic simulations along with market data replications using well studied methods. Notably, the literature relative to their calibration is significant. Now, such models are widely used and well integrated in the companies processes and though their limitations are more understood, the inertia of market practices makes those models quite inescapable.

[2]Compared to time periods involved in insurance.

Real-World ESG. For investment planning or risk management, forward-looking scenarios could be helpful for insurers. For such considerations, it is no longer the replication of market prices that is desired but that of some properties of empirical distributions. Namely, the historical returns of the assets should be duplicated by generated scenarios. In addition to historical targets, some targets can be *motivated by economic forecasts*. For instance, anticipations on average level of inflation rates may be taken into account. Models for real-world scenarios are essentially composed of statistical models of time series. Calibrations methods are thus based on statistical techniques that involve large historical data set.

Real-World ESGs are also central for regulatory computations in Solvency II. Indeed when computing the *Solvency Capital Requirement* or when projecting it on future date (Pillar II below), it is necessary to simulate the state of the economy under the Real-World measure and thus generate paths reproducing some stylized facts. Based on those historical paths, Risk-Neutral computations are performed in a second phase to value the liability side of the undertaking from a Risk-Neutral point of view and eventually derive the projected values of the *Best Estimates*. Real-World modelling is thus pivotal also for regulatory compliance.

ESG for regulatory compliance. For solvency requirements, the Solvency II legislation proposes to value the balance sheet of the insurance companies (and reinsurance) from a *financial* perspective. Relevant ESGs for such computations are thus *Risk-Neutral*. To be used for such regulatory calculations, ESGs should have few properties that can be found in [4] in accordance with [20] and summarized in [9] for the Euro zone or in [23] for broader context. Note that the following requirements are not specific to regulatory computations as they may be also wanted for other applications. To respect these requirements, ESGs users must set up monitoring measures; some are quite common, other can be specific to a company. Be that as it may, companies have to report and justify to the authorities the methodological choices and underlying assumptions made when using their ESG.

ESGs are asked to replicate the regulatory risk-free yield curve at date when the computation is performed. This is motivated by fact that the whole methodology to value the balance sheet is based on risk-free discounts of cash-flows. The derivation of the regulatory risk-free term structure is not straightforward as market rates illustrate equilibrium on financial market resulting from considerations of financial agents whose concerns differ from that of insurance companies. The methodology followed by the European regulator, the European Insurance and Occupational Pensions Authority (EIOPA), to build the regulatory curve is described notably in [20]. The ability of replicating the initial term-structure motivates the choice of the dedicated interest rates model.

Models are also asked to satisfy the Non-Arbitrage Opportunity (NAO) assumption. Arbitrage opportunities are considered as rare events on market and thus not desirable in ESGs that aim to simulate economies. To check the NAO assumption is valid on simulated paths, some *martingale tests* are performed: empirical means computed on simulations are compared to theoretical expectations of some martingale quantities. The tests pass when the two are close enough (a tolerance threshold is given based on statistical sampling error). Note also that NAO assumption implies that interest rates model should perfectly fit the initial interest rates curve. The replication is also checked to validate the used ESG is consistent with NAO assumption.

ESGs have to accurately replicate market prices: this is the so-called *market consistency*. This criterion is motivated by fact that "*the calculation of technical provisions shall make use of and be consistent with information provided by the financial markets and generally available data on underwriting risks (market consistency)*", according to [1]. This notion and its consequences are discussed in [24]. Consequently, ESGs have to be accurately calibrated to market prices thus relying on appropriate calibration methods and accurately simulated so that Monte-Carlo prices are close to market ones. Several metrics are usually given to assess the accuracy at different stages (distance between market and model prices obtained as outputs of the calibration, distance between market and Monte-Carlo prices, and distance between model and Monte-Carlo prices).

An important aspect of ESGs is their ability to jointly simulate economic drivers. The correlation between simulated paths is of interest for insurers. Correlations targets should beforehand be estimated in a model-free approach. Once the models are chosen, those correlations are translated in terms of correlations between risk drivers (i.e. Brownian motions). Empirical and target correlations are then checked to be close enough.

All those tests are though as ensuring the quality of the simulated paths that will be used for computing the Solvency Capital Requirements. Note that some of these are *Monte-Carlo* tests, in the sense that they are based on empirical estimations justified by the Law of large numbers and Central Limit Theorem. It is thus required to simulate a sufficient number of paths to ensure the convergence of empirical quantities and thus an accurate computation of the SCR. [9] discussed the fact that a number of 1000 paths is a minimum[3]. If some tests still fail, some "reprocessing" may be performed either on input data or on simulated paths (cap, floor, removal of atypical data, etc.) but it should remained exceptional and have to be fully justified. Regarding the data used to calibrate the model, they must be chosen accordingly with the risk profile of the company as much as possible. It is quite straightforward for companies to choose which indices must be replicated. However, for interest rates risk, it is sometimes hard to determine what data should be used (we discuss it below). All in all, a large number (several hundreds) of market data should be replicated explaining why calibration is challenging: not only it has to be accurate, but also computationally efficient. The accuracy of the calibration is required so that the simulations of the strategy of the company generated by the ALM model using the paths of the calibrated ESG are consistent with current economic condition observed on market despite the restricted number of possible simulations.

In practice, the number of required calibrations can go from around ten (Standard Formula) to several hundreds (Internal modelling approach).We discuss in Section 3 the problem of the time efficiency of the calibration of the so-called DDSVLMM.

3. Focus on the interest rates modelling in Risk-Neutral environment

A crucial common point of all the presented usages of ESGs is that a today's economic valuation of future cash-flows is performed at some point. The ability of discounting those cash-flows is thus necessary. Moreover, the composition of the portfolio of assets of the companies is so that the interest rates risk is prominent. This is how the interest rates modelling is at the core of the insurance activity. As already mentioned, the most popular models in Risk-Neutral ESGs come from bank industry, in particular those dedicated to interest rates modelling. The choice of the interest rates model embedded in the ESG is led by several factors:

- the accuracy with which the market data are replicated to respect the *Market-Consistency criterion*: swaptions (i.e. call/put options on swap rates) prices are asked to be replicated due to their high liquidity and the fact that they are deemed as capturing the correlation structure between forward rates of different maturities;

- the computational time required by the calibration process: in most of the calibration procedures, interest rates models should replicate more than two hundreds market prices (or volatilities);

[3]This number of scenarios may seem as too small to ensure accurate explorations of the possible state of the world. However, due to complexity of the models used by insurers to compute the sequence of cash-flows associated with the insurance policies they have issued, operational constraints impose that the number of paths should be at most a few thousands.

- the characteristics of the generated paths: e.g., it has lately become necessary to be able to simulate negative interest rates.

A common model among insurers is the Displaced Diffusion with Stochastic Volatility LIBOR Market Model (DDSVLMM): it is a variant of the standard LIBOR Market Model (LMM) comprising a displacement factor and a stochastic volatility. In the standard LMM, forward rates are modelled by log-normal type dynamics under proper probability measure. Adding a displacement coefficient allows to generate negative rates. The stochastic volatility accounts for a better replication of market prices notably of Away-From-The-Money options. The DDSVLMM is an Heston type model that took advantage of the analytic knowledge of its characterisc function. In Section 3.1, we present the standard parametrization of the DDSVLMM. In Section 3.2, we provide an approximation of this standard parametrization that allows alternative pricing method based on Gram–Charlier expansion.

3.1. The DDSVLMM

Let $P(t,T)$ be the time-t price of a Zero-Coupon bond maturing at time $T>t$ with par value 1. Let us consider a finite tenor structure $0 \le T_1 \le T_2 \le \cdots \le T_N$ and denote by $\Delta T_j = T_{j+1} - T_j$. Let us define the forward rate of maturity T_k prevailing over the period $[T_k, T_{k+1}]$ seen at time t is defined by

$$F_k(t) = \frac{1}{T_{k+1} - T_k}\left(\frac{P(t,T_k)}{P(t,T_{k+1})} - 1\right) \tag{3.1}$$

for $t \le T_k$. For $(m,n) \in [\![1,N]\!]^2$, $m \le n$, the swap rate seen at time $t \le T_m$ that prevails over the period $[T_m, T_n]$, can be expressed as

$$S_t^{m,n} = \frac{P(t,T_m) - P(t,T_n)}{\sum_{j=m}^{n-1} \Delta T_j P(t,T_{j+1})} \tag{3.2}$$

according to an arbitrage-free reasoning (see Section 1.5 in [14]). We denote by $B^S(t) := \sum_{j=m}^{n-1} \Delta T_j P(t,T_{j+1})$ the *annuity* of the swap rate. Under the probability measure $\mathbb{P}^S$ (the *forward swap measure*, named after [22]) associated to the *numéraire* B^S, the swap rate is a martingale. In the standard model, the evolution of the swap rate (3.2) through time is described by the following dynamics: for $t \le T_m$,

$$\begin{aligned} \mathrm{d}S_t^{m,n} &= \sqrt{V_t}\boldsymbol{\lambda}^{m,n}(t) \cdot \mathrm{d}\boldsymbol{W}_t^S, \\ \mathrm{d}V_t &= \kappa(\theta - \xi_0(t)V_t)\mathrm{d}t + \epsilon\sqrt{V_t}\mathrm{d}W_t^S, \end{aligned} \tag{3.3}$$

where $(\boldsymbol{W}_t^S)_{0 \le t \le T_m}$ and $(W_t)_{0 \le t \le T_m}$ are respectively D-dimensional and 1-dimensional Brownian motions under $\mathbb{P}^S$. The components of $\boldsymbol{W}^S$ are all independent one another and of W. The following functions are introduced:

$$\boldsymbol{\lambda}^{m,n}(t) := \sum_{j=m}^{n-1} \omega_j(0)\boldsymbol{\gamma}_j(t),\ \xi_0(t) := 1 + \frac{\epsilon}{\kappa}\sum_{j=m}^{n-1}\alpha_j(0)\sum_{k=1}^{j}\frac{\Delta T_k(F_k(0)+\delta)}{1+\Delta T_k F_k(0)}\rho_k(t)\|\boldsymbol{\gamma}_k(t)\|$$

where the coefficients ω_j are defined by

$$\omega_j(0) := \frac{\Delta T_j P(0,T_{j+1})}{B^S(0)}\left(1 + \frac{\Delta T_j}{1+\Delta T_j F_j(0)}\sum_{\ell=m}^{j-1}\left(F_\ell(0) - S_0^{m,n}\right)\right)(F_j(0)+\delta)$$

and the shape of the vector functions $t \mapsto \boldsymbol{\gamma}_j(t)$ are left free to the user. A practical setting is to consider that all those time dependent quantities to be piecewise constant on the grid $[T_j, T_{j+1}[$, $j = 1, \ldots, m$. Note that the derivation of those functions $\boldsymbol{\lambda}^{m,n}$ and ξ_0 follows from Itô's lemma as it is observed from the definition (3.2) that the swap rate is a function of the forward rates. We refer to [25] for details of the computation. The stochastic process $(V_t)_{t\geq 0}$ is a Cox–Ingersoll–Ross process that has been introduced to model the variance process. The coefficients κ, θ, and ϵ are non-negative parameters. They are assumed to satisfy Feller condition $2\kappa\theta \geq \epsilon^2$ that ensures the process V to remain non-negative through time almost surely as long as $V_0 > 0$.

In this parametrization, the process defined by dynamics (3.3) belongs to the class of affine process for which analytical knowledge of the moment generating function is known, through resolution of some Riccati equations. The details of the computation can be found in [19]. Semi-analytical swaptions prices can then be derived based on the integration of the characteristic function of the shifted swap rate. In the following, the moment generating function of the swap rate process is defined over its definition domain $\mathcal{D} \subset \mathbb{C}$ as

$$\Psi(x; t, S_t^{m,n}, V_t) := \mathbb{E}^S\Big[e^{x\left(S_{T_m}^{m,n}+\delta\right)}|\mathcal{F}_t\Big],\ x \in \mathcal{D},$$

where $\mathcal{F}_t$ is the market information known at time t (filtration generated by the Brownian motions representing the risk on the market). In the following proposition, the time$-t$ price of a swaption on swap rate of maturity T_m and tenor T_n and of strike K is denoted by

$$PS(t, T_m, T_n, K) := B^S(0)\mathbb{E}^S\big[\max(S_{T_m}^{m,n} - K, 0)\big].$$

Proposition 3.1 (Normal swaption pricing under DDSVLMM)**.** *In the DDSVLMM, swaption price expresses as*

$$PS(t, T_m, T_n, K) = B^S(0)(S_0^{m,n}P_1 - KP_2),$$

where

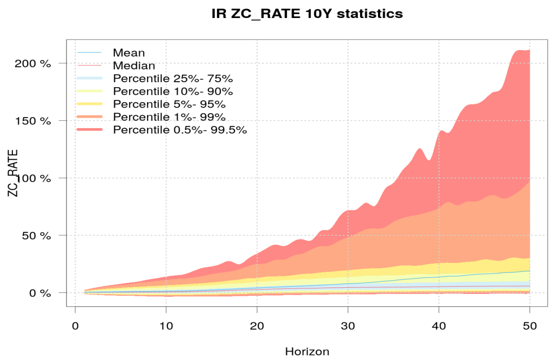
$$P_1 = \frac{1}{2}S_0^{m,n} - \frac{1}{\pi}\int_0^{+\infty} \mathrm{Im}\left(\frac{\Psi'(-iu; 0, s_0^{m,n}, v_0)e^{iuK}}{u}\right)\mathrm{d}u,$$
$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathrm{Im}\left(\frac{e^{-iuK}\Psi(iu; 0, s_0^{m,n}, v_0)}{u}\right)\mathrm{d}u.$$

The main issue in this model when it comes to price several hundred of options for calibrating the dynamics (3.3) is that an important number of quadratures are necessary to numerically compute the integrals in Proposition 3.1. This makes the full calibration time quite long and researches have been led in order to reduce it: in [19], authors proposed to approximated swaptions prices using Gram–Charlier density approximation techniques or in [18] in which authors worked in an equity-type context and derive the expression of the analytical gradient of the prices to input in gradient-based optimization algorithms. More recently, it has been proposed to further developp the Gram–Charlier approximation technique in this framework of the DDSVLMM in which the convergence of the expended series is ensured.

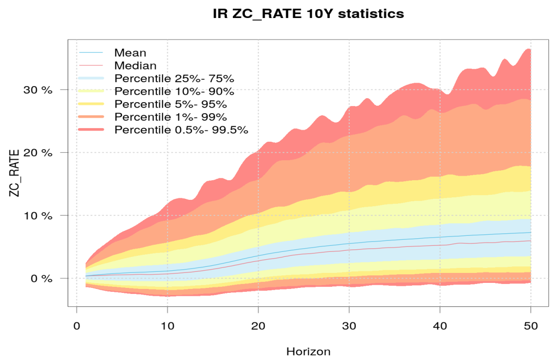
A practical point of view

To compute insurance quantities (SCR, BE, NAV, etc.), we have to simulate paths of all the financial risk drivers that have been selected to describe the modelled economy: interest rates, equities, real-estate, real rates/inflation and credit spreads are usually modelled. To obtain these paths, we usually discretize the stochastic differential equations defining the models dedicated to each risk drivers. When possible, we resort to exact simulations methods. This can be a tricky task for the more elaborated models, such as the DDSVLMM since the simulation of CIR process is complex but well studied in the literature.

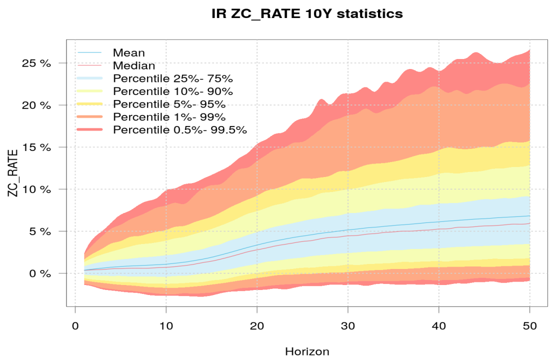
We provide below some elements of analysis on the impact of the parameters defining the DDSVLMM on the computations of insurance quantities. The values itself of insurance quantities are key for insurers but the associated uncertainty on it is also an important information, in view of the amounts that are involved. This uncertainty is of course impacted by the explosiveness



(A) $(\kappa, \rho) = (0.1, 0.8)$.



(B) $(\kappa, \rho) = (0.229, 0.9)$.



(C) $(\kappa, \rho) = (0.4, 0.998)$.

FIGURE 3.1. 10Y Zero-Coupon rate quantiles.

embedded in generated paths. As explained above, insurers can not increase the number of simulations to reduce it due to limited operational resources.

To illustrate these considerations, we change the value of the parameter monitoring the mean reversion speed and the correlation rates-volatility, respectively denoted by κ and ρ above. We provide, for each case, (i) quantiles trajectories of 10Y Zero-Coupon rate and (ii) the BE estimation. The quantiles trajectories allow to visualize the deformation of distributions of simulated risk factors through time. This information is easily extracted from the set of trajectories. In Figure 3.1, quantiles paths of 10Y Zero-Coupon rate for the pairs $(\kappa, \rho) \in (0.1, 0.8), (0.229, 0.9)$ or $(0.4, 0.998)$. Note that $(\kappa, \rho) = (0.229, 0.998)$ is obtained after a calibration on 12/31/2021 market data; others parameters are fixed to their values obtained as outputs of the calibration.

We provide in Table 3.1 the sensitivity of the BE estimation. This information is more complex to have in practice, since it involves the use of ALM models, which is heavy from a computational point of view. We observe a variation of the BE up to 350k€ depending on the parameters which is not negligible. The amounts of cash that are involved in such computations explain the need of accurate and robust calibrations of the models that composed the ESG (see Section 2.4). Some comments to explain the results in Table 3.1: the case $(\kappa, \rho) = (0.1, 0.8)$ generates the most explosive interest rates paths (see Figure 3.1); in the simplified used ALM model, a rise in the interest rates will be in favour of the company as it will allow the insurer to increase its incomes by earning the spread between market rate and fixed delivered rate to the policyholder, without being impacted by policyholder's behaviour. The insurer is then able to reduce its liabilities. This is not really realistic as if such a case appears in reality policyholders would massively buyout if insurer does not increase the delivered rate. Uncertainty on the value of the BE (semi

Table 3.1. Sensitivity of the BE estimation to the long-term level of stochastic volatility of interest rates. All quantities are expressed in M€.

(κ, ρ)	Estimated BE value	Semi 95% confidence interval
(0.1,0.8)	958.03	$\pm$ 5.90
(0.229,0.9)[4]	958.26	$\pm$ 5.51
(0.4,0.998)[5]	958.35	$\pm$ 5.25

confidence interval) also vary significantly following the parametrization; here, the uncertainty increases with the explosiveness of the rates. In practice, when using more elaborated ALM models, we observe even more pronounced impacts on the BE.

To obtain a value for the SCR (amount of cash to be immobilized), an important number of intermediary quantities (BE, NAV, etc.) should be computed beforehand, as explained in Section 2.2. In particular, they are computed based on observed market data and stressed data (shift of the initial interest rates curve, shocks on implied volatilities of derivatives, etc.). For all stressed scenarios, calibrations followed by the simulations of models composing the ESG are necessary. Hence the necessity of falling back on efficient calibration methods, notably for the models dedicated to interest rates which are usually the more delicate to calibrate. We present in the following section an innovative method to calibrate (3.3) based on series expansion, extending the works of [19] and [5].

3.2. **Jacobi process in the DDSVLMM**

It has been introduced in [8] the *Jacobi version* of the DDSVLMM which is an approximation of the model (3.3). In the proposed setting, the volatility factor is modelled by a $[v_{\min}, v_{\max}]$-valued Jacobi process. The swap rate modelled in our proposal is denoted by $S^{m,n,J}$ and is described over $[0, T_m]$ by the following stochastic differential equation

$$\begin{aligned} \mathrm{d}S_t^{m,n,J} &= \sqrt{Q(V_t)}\rho(t)\|\boldsymbol{\lambda}^{m,n}(t)\|\mathrm{d}W_t + \sqrt{V_t - \rho(t)^2 Q(V_t)}\boldsymbol{\lambda}^{m,n}(t)\cdot \mathrm{d}\boldsymbol{W}_t^{S,*}, \\ \mathrm{d}V_t &= \kappa(\theta - \xi^0(t)V_t)\mathrm{d}t + \epsilon\sqrt{Q(V_t)}\mathrm{d}W_t, \end{aligned} \quad t \leq T_m \tag{3.4}$$

where $\boldsymbol{\lambda}^{m,n}$, ξ^0 and ρ are defined as in (3.3), Q is a bounding function defined by $Q(v) = \frac{(v_{\max}-v)(v-v_{\min})}{(\sqrt{v_{\max}}-\sqrt{v_{\min}})^2}$, where $0 \leq v_{\min} < v_{\max} \leq \infty$. Observe that $Q(v) \leq v$ for any $v \in \mathbb{R}$ and that $Q(v) \geq 0$ for $v \in [v_{\min}, v_{\max}]$. We recall that all components of $\boldsymbol{W}^{S,*}$ are independent and are also all independent from W. The volatility factor $(V_t)_{t\geq 0}$ follows a Jacobi dynamics with additional time dependency in the drift. For this dynamics, the Feller condition writes:

$$\frac{\epsilon^2(v_{\max} - v_{\min})}{(\sqrt{v_{\max}} - \sqrt{v_{\min}})^2} \leq 2\kappa \min\left(\xi^0_{\min} v_{\max} - \theta, \theta - \xi^0_{\max} v_{\min}\right). \tag{3.5}$$

where we have introduced the bounds $\xi^0_{\min} \leq \xi^0(\cdot) \leq \xi^0_{\max}$. It ensures the process V in (3.4) to remain bounded at any date: $\mathbb{P}^S(\forall t \in [0, T_m] : V_t \in (v_{\min}, v_{\max})) = 1$. In this setting, the coefficient $\rho(t)$ is interpreted in dynamics (3.4) as a scaling factor of the instantaneous correlation between the swap rate and its volatility since the following holds:

$$\frac{\mathrm{d}\left\langle V_\cdot, S_\cdot^{m,n,J}\right\rangle_t}{\sqrt{\mathrm{d}\left\langle V_\cdot, V_\cdot\right\rangle_t}\sqrt{\mathrm{d}\left\langle S_\cdot^{m,n,J}, S_\cdot^{m,n,J}\right\rangle_t}} = \rho(t)\sqrt{\frac{Q(V_t)}{V_t}}.$$

[5]Calibrated value of κ on 12/31/2021 market data. Used for plotting Figure 2.2.

[5]Calibrated value of ρ on 12/31/2021 market data. Used for plotting Figure 2.2.

Since $0 \leq Q(v) \leq v$ for $v \in [v_{\min}, v_{\max}]$, the instantaneous correlation is smaller than $\rho(t)$ at each time. Observing that $Q(v) \to v$ when $(v_{\min}, v_{\max}) \to (0, +\infty)$, it can be shown that the swap rate process defined by (3.4) converges weakly (and strongly) towards the process defined in (3.3) (see [8]); thus, present framework (3.4) can be seen as a numerical approximation of (3.3).

The fact that the volatility process remains bounded through time allows to write swaptions prices as convergent series whose coefficient are linear combinations of the moments of the swap rates. To compute its moments, the polynomial property of the swap rate is now exploited: the characteristic function is no longer known but the sequence of moments can be derived through matrix exponentiation (see [17] for an introduction to the theory of polynomial processes). Those convergent series are known in the literature as Gram–Charlier series: they are based on an expansion technique that amounts to approximate an unknown density with a Gaussian reference one. It is defined notably in [16][6]. Here, we aim at approximating the density of the swap rate process at a fixed date. Let us denote by g_r the auxiliary reference density (Gaussian) and by g_T the density of the random variable $S_T^{m,n}$ defined by dynamics (3.3). To present the Gram–Charlier expansion technique, let us introduce the Hilbert space of squared-integrable function with respect to g_r:

$$\mathcal{L}_r^2 = \left\{ h : \mathbb{R} \to \mathbb{R} \text{ measurable such that } \|h\|_r^2 := \int_{\mathbb{R}} h(u)^2 g_r(u) \mathrm{d}u < \infty \right\}.$$

When the likelihood ratio g_T/g_r lies in $\mathcal{L}_r^2$, the properties of this Hilbert space allows to write swaption price as a convergent serie:

$$\begin{aligned} PS(T, T_m, T_n, K) &= B^S(0) \int_{\mathbb{R}} \max(x - K, 0) g_T(x) \mathrm{d}x \\ &= B^S(0) \int_{\mathbb{R}} \max(x - K, 0) \frac{g_T(x)}{g_r(x)} g_r(x) \mathrm{d}x \\ &= B^S(0) \sum_{n \geq 0} \phi_n h_n, \end{aligned} \tag{3.6}$$

where $\phi_n = \int_{\mathbb{R}} \max(x - K, 0) H_n(x) g_r(x) \mathrm{d}x$, $h_n = \int_{\mathbb{R}} H_n(x) g_T(x) \mathrm{d}x = \mathbb{E}^S[H_n(S_T^{m,n})]$ and the sequence $(H_n)_{n \in \mathbb{N}}$ forms an orthonormal basis of polynomials of $\mathcal{L}_r^2$. In present case of Hilbert space built on Gaussian density, those polynomials are known as the Hermite polynomials. Observe then that the coefficients h_n write as linear combinations of the moments of $S_T^{m,n}$.

To ensure the likelihood ratio to lie in $\mathcal{L}_r^2$ and thus obtain the equality (3.6), the following result illustrates why the bounding assumption on the volatility process is crucial. In the following result, $\lambda_{\max} = \sup_{t \leq T} \|\boldsymbol{\lambda}^{\boldsymbol{m},\boldsymbol{n}}(t)\|$; recall this is a finite quantity as we have assumed $t \mapsto \|\boldsymbol{\lambda}^{\boldsymbol{m},\boldsymbol{n}}(t)\|$ is piecewise constant.

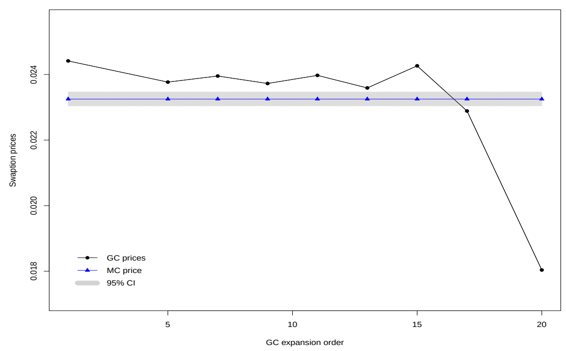
Theorem 3.2. *Suppose that $4\kappa\theta > \epsilon^2$, $\sup_{t \leq T} |\rho(t)| < 1$, $v_{\min} \geq 0$ and $v_{\max} < \infty$. Consider g_r is centred Gaussian density of variance σ_r^2 satisfying*

$$\sigma_r^2 > \frac{T v_{\max}}{2} \lambda_{\max}^2. \tag{3.7}$$

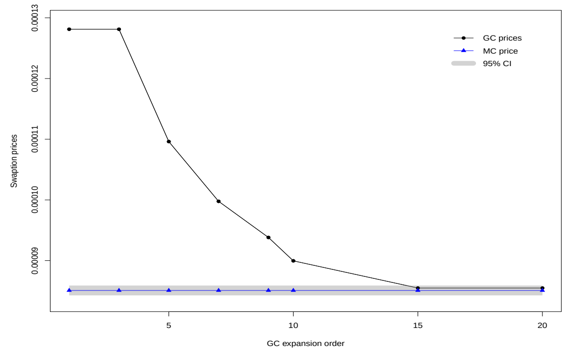
Then, a Gram–Charlier expansion can be performed on the density g_T of (3.4) using g_r the reference density. In particular, the sufficient condition to the $\mathcal{L}_r^2-$convergence of the family of approximating densities is satisfied; that is

$$\int_{\mathbb{R}} \frac{g_T(u)^2}{g_r(u)} \mathrm{d}u < \infty.$$

[6]Historically speaking, the Gram–Charlier technique refers to pointwise expansion of the unknown density using a perturbation of a Gaussian density as reference, see notably [16]. By abuse of terms, we will still speak of Gram–Charlier expansion here when working in an Hilbert space built around a Gaussian density.



(A) $\sigma_r = \sqrt{\mathrm{Var}^S(S_T^{m,n})} = 1.273 \times 10^{-2}$, $\lambda^2_{\max} = 1.648 \times 10^{-4}$, $v_{\max} = 0.5$. Maturity and tenor of the swaption: $T = T_m = 5$ and $T_n - T_m = 5$.



(B) $3.203 \times 10^{-4} = \sigma_r \neq \sqrt{\mathrm{Var}^S(S_T^{m,n})}$, $\lambda^2_{\max} = 8.097 \times 10^{-7}$, $v_{\max} = 0.25$. Maturity and tenor of the swaption: $T = T_m = 1$ and $T_n - T_m = 1$.

FIGURE 3.2. Expanded Gram–Charlier prices compared to Monte-Carlo prices (computed on 10^5 simulations. Monte-Carlo prices are using after discretizing the original dynamics 3.4.

While being a sufficient condition, (3.7) is sharp: a “slight” non-satisfaction of it can cause a divergence of the expansion series and thus a bad approximation of swaptions prices, as illustrated. In Figure 3.2, we computed expanded prices by working in Hilbert spaces associated with different auxiliary variance parameters σ_r, including a case for which the sufficient condition (3.7) is not satisfied (Figure 3.2 (a))

A practical point of view

Writing swaptions prices as truncated series allows to employ closed-form formulas to derive approximated swaptions prices while in the standard modelling framework, one has to approximate numerically integrals appearing in swaptions prices. Truncating the series induces an error that can be reduced by a wise choice of the auxiliary variance parameter. By doing so, the series involved in (3.6) can be made quickly converging so that a restricted number of moments of $S_T^{m,n}$ are required in practice. We illustrate this point below in Figure 3.3 in which from the very first expansion order, Gram–Charlier prices reach the 95% confidence interval built around the Monte-Carlo estimation of the price (10^5 simulations) coming from discretization of (3.4).

This can lead to a substantial reduction of the computational time required to obtain swaptions prices compared to the more common model (3.3). To choose the auxiliary parameter to get a fast convergence of the expanded series, one should take $\sigma_r^2 = \mathrm{Var}(S_T^{m,n})$: the underlying idea is that the more g_r and g_T have moments in common, that faster the series will converge. However, such a choice usually do not allow to satisfy constraint (3.7). A practical solution would be to set $\sigma_r^2 = \frac{Tv_{\max}}{2}\lambda^2_{\max} + \eta$ with η being a very small number. This way, we are able to calibrate (3.4) while ensuring the closeness of prices induced by the model and prices computed on generated simulations.

We provide in the table 3.2 the normalized approximated computational times required to calibrate the DDSVLMM on 336 swaptions prices (quoted on EURO market on the 30/06/2020). The calibration of (3.3) based on Gaussian quadrature is the reference calibration method. We also perform a calibration of (3.4) using a 4^{th} order truncation of the expanded series as explained above. This allows a substantial reduction of the computational time required to compute the interest rates model compared to the Gaussian quadrature based method since the time has been divided by 2. We also have tested to integrate the gradient of the objective function in the optimization algorithm, either analytically (for the Gaussian quadrature in (3.3) or numerically

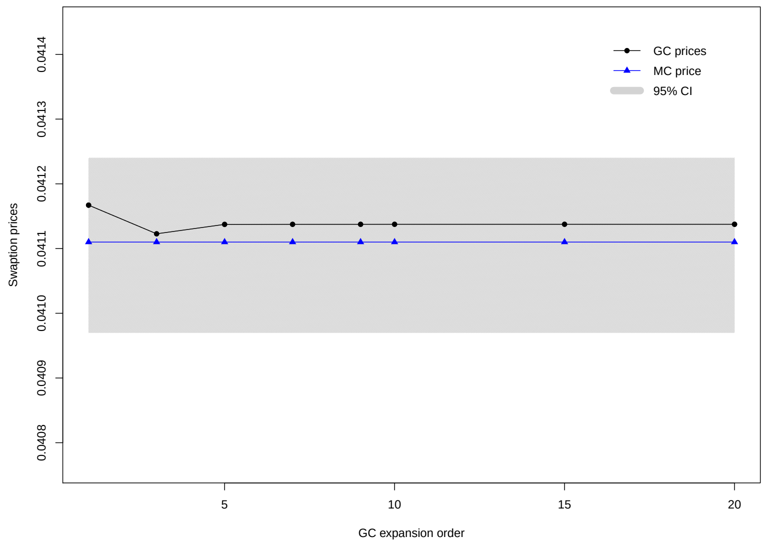


Figure 3.3. Fast convergence of expanded Gram–Charlier prices with $\sigma_r^2 = \mathrm{Var}(S_T^{m,n}) = 4.025 \times 10^{-5}$, $\lambda_{\max} = 1.411 \times 10^{-4}$, $v_{\max} = 0.05$, $T_m = 10$, $T_n - T_m = 4$, $v_0 = 0.015$.

(for the Gram–Charlier expansion of (3.4)). For the latter, we observe that the computational time represents only 5% of the reference time.

Table 3.2. Computational times required for calibrating the DDSVLMM.

Method	Normalized CPU times
Gaussian quadrature in (3.3)	~ 100
Analytical gradient in (3.3)	~ 20
Gram–Charlier in (3.4)	~ 50
Numerical gradient in expanded series of (3.4)	~ 5

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