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Parasite infection in a cell population with deaths and reinfections

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Abstract

We introduce a model of parasite infection in a cell population, where cells can be infected, either at birth through maternal transmission, from a contact with the parasites reservoir, or because of the parasites released in the cell medium by infected cells. Inside the cells and between infection events, the quantity of parasites evolves as a general non linear branching process. We study the long time behaviour of the infection.

Introduction

The aim of this paper is to complement the study of parasite or viral infection in a cell population conducted in [7, 9, 22] by adding possible reinfections, either due to new contacts of the cell population with the reservoir of the pathogen, or within the cell population, due to the pathogens released in the medium by infected cells.

Many pathogens, in particular the ones responsible for emerging infectious diseases, are transmitted both between and within a host community. The Oxford Textbook of Global Public Health [12] defines the reservoir of a pathogen as “any person, other living organism, or inanimate material in which the infectious agent normally lives and grows”. The epidemiological dynamics of infectious diseases is highly dependent on the transmission from the reservoir [31], and the start of an outbreak is promoted by a primary contact between the reservoir and the incidental host (i.e. a host that becomes infected but is not part of the reservoir) leading to the potential transmission of the infection to the host population. We aim at taking into account such mechanisms.

We will thus assume that a cell population may be recurrently infected by contacts of the cells with a reservoir, and that the pathogens may also be transmitted within the cell population, either when a cell divides into two daughter cells receiving respectively a fraction Θ and $1 - \Theta$ of the pathogens of their mother, or when pathogens are extruded from the cells. In this work, the cell death and division rates do not depend on the quantity of pathogens in the cells, but the quantity of pathogens released in the medium depends on the quantity of pathogens in the cell population. A generalization to death and division rates dependent on the quantity of pathogens in the cell could be the subject of future work. This simplification follows from the difficulty to obtain an explicit spinal process (see later).

We have two examples in mind:

- (1) Bacteriophages M13: Bacteriophages are a class of viruses infecting bacteria. They show applicability in the field of gene delivery [18]. Bacteriophages M13 have been commonly used to display vaccine antigens (for instance hepatitis B mimotopes [25] or Plasmodium

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vivax surface protein mimotopes [3]). Their multiplication begins immediately upon infection [10], and they are continually assembled at, and extruded through, the cell wall into the medium [29]. The host cell is not killed and continues to divide, albeit at a reduced rate [30].

- (2) The class of encapsulated viruses is also a good example for our model. They can cause persistent infection as they usually not kill the cell to be extruded. The high immunological flexibility of envelope proteins allows some enveloped viruses to multiply in different host species. Viruses transmitted by arthropods (e.g. mosquitoes or mites), known as arboviruses, are mostly enveloped viruses. Viruses are particularly pathogenic when they occur in a new host population, where no immunity to them exists. For this reason, enveloped viruses, which are particularly easy to transmit from animals to humans, have a particularly high potential to cause new infections in humans. Examples of enveloped viruses include COVID-19, Influenza, Hepatitis B and C, and Ebola Virus Disease.

The dynamics of the quantity of parasites in a cell will be given by a Stochastic Differential Equation (SDE) with a diffusive term and positive jumps. Then, after an exponential time, the cell dies, or divides and shares its parasites between its two daughter cells. We are interested in the long time behaviour of the parasite infection in the cell population. More precisely, we give bounds on the fraction of cells containing a small, large, finite or infinite quantity of parasites, depending on the assumptions under consideration. We refer to [22] for references on biological studies and on previous mathematical works on the class of processes under study. In particular, references [1, 2, 4, 5, 17, 24, 26] focused on branching within branching to study host-parasites dynamics. Similarly as in [6, 11, 14, 15, 20, 22], the proof strategy consists in introducing a spinal process, and investigating the long time behaviour of this process that corresponds to the trait of a uniformly sampled individual in the population. Via a Many-to-One formula (see Proposition 1.2), we may deduce properties on the long time behaviour of the process at the population level. Even if the cell division rate does not depend on the quantity of parasites, the spinal process may be non-homogeneous, because of the reinfection mechanism.

The model considered here is close to a recent paper, in collaboration with Aline Marguet [22]. Some parts of the dynamics are simplified, as cell birth and death rates do not depend on the quantity of parasites in the cell, and we do not allow stable jumps. The novelty of the current model is to take into account the possibility of reinfections by the reservoir and the parasites released in the cell medium by infected cells, and to study the impacts of these two ways of reinfection on the long term behaviour of the infection. We will use results stated in a previous paper with the same coauthor [21] to derive properties of the auxiliary process, in particular its behaviour close to infinity (see Conditions (SN ∞) and (LN ∞) on page 28).

In the sequel $\mathbb{N} := \{0, 1, 2, \dots\}$ will denote the set of nonnegative integers, $\mathbb{R}_+ := [0, \infty)$ the real line, $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ and $\mathbb{R}_+^* := (0, \infty)$. We will denote by $\mathcal{C}_b^2(A)$ the set of twice continuously differentiable bounded functions on a set A . Finally, for any stochastic process X on $\bar{\mathbb{R}}_+$ or Z on the set of point measures on $\bar{\mathbb{R}}_+$, and any real function f on \mathbb{R}_+ , we will denote by $\mathbb{E}_x[f(X_t)] = \mathbb{E}[f(X_t)|X_0 = x]$, $\mathbb{E}[f(X_t)] = \mathbb{E}_0[f(X_t)]$ and $\mathbb{E}_{\delta_x}[f(Z_t)] = \mathbb{E}[f(Z_t)|Z_0 = \delta_x]$.

1. Definition of the population process

1.1. Parasites dynamics in a cell

Each cell contains parasites whose quantity evolves as a diffusion with positive jumps. More precisely, we consider the SDE

$$\mathfrak{X}_t = x + \int_0^t g(\mathfrak{X}_s) ds + \int_0^t \sqrt{2\sigma^2(\mathfrak{X}_s)} dB_s + \int_0^t \int_0^{p(\mathfrak{X}_{s-})} \int_{\mathbb{R}_+} z \tilde{Q}(ds, dx, dz), \quad (1.1)$$

where x is nonnegative, g , $\sigma \geq 0$ and $p \geq 0$ are real functions on \mathbb{R}_+ , B is a standard Brownian motion, Q is a Poisson Point measure with intensity $ds \otimes dx \otimes \pi(dz)$, \tilde{Q} the corresponding compensated measure, and π is a nonnegative measure on \mathbb{R}_+ . Finally, B and Q are independent.

By choosing such a dynamics we aim at obtaining a model as general as possible. This is a generalisation of continuous state branching processes, in which a dependence of growth rates and fluctuations on the state of the population is allowed, in order to model, for example, a competition for resources or space, or a mutualistic interaction (see [19] for the introduction of this kind of processes). The diffusive term allows to take into account moderate size birth events, and the jump term allows to take into account large birth events (see [8] for a characterisation of the scaling limits of Galton–Watson processes with different types of birth events).

Under our conditions (see later), the SDE (1.1) has a unique non-negative strong solution. In this case, it is a Markov process with infinitesimal generator \mathcal{G} , satisfying for all $f \in C_b^2(\overline{\mathbb{R}_+})$,

$$\mathcal{G}f(x) = g(x)f'(x) + \sigma^2(x)f''(x) + p(x) \int_{\mathbb{R}_+} (f(x+z) - f(x) - zf'(x)) \pi(dz), \quad (1.2)$$

and 0 and $+\infty$ are two absorbing states. Following [20], we denote by $(\Phi(x, s, t), s \leq t)$ the corresponding stochastic flow i.e. the unique strong solution to (1.1) satisfying $\mathfrak{X}_s = x$ and the dynamics of the trait between division events is well-defined.

1.2. Cell division and death

Each cell divides at rate b and is replaced by two cells with quantity of parasites at birth given by Θx and $(1 - \Theta)x$. Here Θ is a nonnegative random variable on $(0, 1)$ with symmetric distribution $\kappa(d\theta)$ (that is to say $\int_0^1 f(\theta)\kappa(d\theta) = \int_0^1 f(1 - \theta)\kappa(d\theta)$ for any function f continuous on $[0, 1]$). Finally, each cell dies at rate d .

1.3. Reinfection

New parasites can enter cells through two distinct mechanisms:

- either the population comes into contact with the reservoir of the parasite and each cell is infected at a rate $\lambda(\cdot)$ by a quantity of parasites \mathfrak{i} that follows a law \mathfrak{J} . This rate may depend on the quantity of parasites in the cell. An increasing rate with the quantity of parasites could indicate that the cell's defences are weakened or that the parasite is altering the properties of its host to facilitate infection by new parasites. A decreasing rate could mean that the immune system has detected the parasite and is taking measures to limit the infection;
- or the cells are infected by parasites released by other cells in the medium. The rate of infection at time t (parametrised by the function $r(\cdot)$) then depends on the mean quantity of parasites present in the cell population at time t divided by the mean number of cells at time t , $e^{(b-d)t}$. Ideally, it would be convenient to make the reinfection rate dependent on the quantity of parasites and not its expectation, but the equations obtained would not be closed and the dynamics of the infection could not be studied (see Remark 1.3). The quantity \mathfrak{p} of parasites entering a cell during such an infection event follows a law \mathfrak{P} .

1.4. Existence and uniqueness

We use the classical Ulam–Harris–Neveu notation to identify each individual. Let us denote by

$$\mathcal{U} := \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$$

the set of possible labels, $\mathcal{M}_P(\mathbb{R}_+)$ the set of point measures on \mathbb{R}_+ , and $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_P(\mathbb{R}_+))$, the set of càdlàg measure-valued processes. For any $Z \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_P(\mathbb{R}_+))$, $t \geq 0$, we write

$$Z_t = \sum_{u \in V_t} \delta_{X_t^u}, \quad (1.3)$$

where $V_t \subset \mathcal{U}$ denotes the set of individuals alive at time t and X_t^u the trait at time t of the individual u . We also write, for a function f on \mathbb{R}_+ ,

$$\langle f, Z_t \rangle = \sum_{u \in V_t} f(X_t^u).$$

Let $E = \mathcal{U} \times (0, 1) \times \mathbb{R}_+$, $E_{r_1} = E_{r_2} = \mathcal{U} \times \mathbb{R}_+ \times \mathbb{R}_+$ and $M(ds, du, d\theta, dz)$, $M_{r_1}(ds, du, di, dz)$, $M_{r_2}(ds, du, dp, dz)$ be independent Poisson Point measures on $\mathbb{R}_+ \times E$, $\mathbb{R}_+ \times E_{r_1}$, $\mathbb{R}_+ \times E_{r_2}$ with respective intensity $ds \times n(du) \times \kappa(d\theta) \times dz$, $ds \times n(du) \times \mathfrak{I}(di) \times dz$, $ds \times n(du) \times \mathfrak{P}(dp) \times dz$, where $n(du)$ denotes the counting measure on \mathcal{U} . Let $(\Phi^u(x, s, t), u \in \mathcal{U}, x \in \bar{\mathbb{R}}_+, s \leq t)$ be a family of independent stochastic flows satisfying (1.1) describing the individual-based dynamics. We assume that M , M_{r_1} , M_{r_2} and $(\Phi^u, u \in \mathcal{U})$ are independent. We denote by $(\mathcal{F}_t, t \geq 0)$ the filtration generated by the Poisson Point measures M , M_{r_1} and M_{r_2} and the family of stochastic processes $(\Phi^u(x, s, t), u \in \mathcal{U}, x \in \bar{\mathbb{R}}_+, s \leq t)$ up to time t .

We now introduce assumptions to ensure the existence and strong uniqueness of the process. Points (i) to (iii) of Assumption EU (for Existence and Uniqueness) ensure that the dynamics in a cell line is well defined between the reinfection times (as the unique nonnegative strong solution to the SDE (1.1) up to explosion, and infinite value of the quantity of parasites after explosion); point (iv) ensures that a cell has a higher probability to be infected when there are more parasites in the cell medium, and that if there is no parasite in the cell population, there is no reinfection.

Assumption EU.

- (i) *The function p is locally Lipschitz on \mathbb{R}_+ , non-decreasing and $p(0) = 0$. The function g is continuous on \mathbb{R}_+ , $g(0) = 0$ and for any $n \in \mathbb{N}$ there exists a finite constant B_n such that for any $0 \leq x \leq y \leq n$*

$$|g(y) - g(x)| \leq B_n \phi(y - x), \quad \text{where } \phi(x) = \begin{cases} x(1 - \ln x) & \text{if } x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

- (ii) *The function σ is Hölder continuous with index $1/2$ on compact sets and $\sigma(0) = 0$.*

- (iii) *The measure π satisfies $\int_0^\infty (z \wedge z^2) \pi(dz) < \infty$.*

- (iv) *r and λ are continuous, r is non-decreasing and bounded, $\int_0^\infty i\mathfrak{I}(di) + \int_0^\infty p\mathfrak{P}(dp) < \infty$, and $r(0) = 0$.*

Recall the definition of \mathcal{G} in (1.2). Then the structured population process may be defined as the strong solution to a SDE.

Proposition 1.1. *Under Assumption EU there exists a strongly unique \mathcal{F}_t -adapted càdlàg process $(Z_t, t \geq 0)$ taking values in $\mathcal{M}_P(\overline{\mathbb{R}}_+)$ such that for all $f \in C_b^2(\overline{\mathbb{R}}_+)$ and $x_0, t \geq 0$,*

$$\begin{aligned} \langle Z_t, f \rangle &= f(x_0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{G}f(x) Z_s(dx) ds + M_t^f(x_0) \\ &+ \int_0^t \int_E \mathbb{1}_{\{u \in V_{s-}\}} \left(\mathbb{1}_{\{z \leq b\}} (f(\theta X_{s-}^u) + f((1-\theta)X_{s-}^u)) - \mathbb{1}_{\{z \leq b+d\}} f(X_{s-}^u) \right) M(ds, du, d\theta, dz) \\ &+ \int_0^t \int_{E_{r_1}} \mathbb{1}_{\{u \in V_{s-}\}} \left(\mathbb{1}_{\{z \leq \lambda(X_{s-}^u)\}} (f(X_{s-}^u + i) - f(X_{s-}^u)) \right) M_{r_1}(ds, du, di, dz) \\ &+ \int_0^t \int_{E_{r_2}} \mathbb{1}_{\{u \in V_{s-}\}} \left(\mathbb{1}_{\{z \leq r(e^{-(b-d)s} \mathbb{E}_{\delta_{x_0}}[\langle Z_s, Id \rangle])\}} (f(X_{s-}^u + p) - f(X_{s-}^u)) \right) M_{r_2}(ds, du, dp, dz), \end{aligned}$$

where for all $x \geq 0$, $M_t^f(x)$ is a \mathcal{F}_t -martingale, and Id denotes the identity function.

We always assume that there is one cell at time 0. $e^{(b-d)s}$ is thus the mean number of cells at time s , and $e^{-(b-d)s} \mathbb{E}[\langle Z_s, Id \rangle]$ is the mean quantity of parasites in the population divided by the mean number of cells at time s .

The proof of Proposition 1.1 is a combination of [27, Proposition 1] and [20, Theorem 2.1]. It is essentially the same as the proof of Proposition 1.1 in [22] and we refer the reader to this paper. The only difference is the addition of the reinfection events. Reinfection events due to the release of parasites in the cell medium have a bounded rate (as r is bounded), thus we can construct the solution between these reinfection events, choose uniformly the cell which undergoes the infection, add the quantity of parasites in the cell, and reiterate the procedure until the next infection event of this type occurs. Due to the dynamic of the number of parasites in the cells (given by \mathcal{G}), the quantity of parasites may reach infinity in a cell u_0 for instance. But in this case,

$$\mathbb{1}_{\{u_0 \in V_{s-}\}} \left(\mathbb{1}_{\{z \leq \lambda(X_{s-}^{u_0})\}} (f(X_{s-}^{u_0} + i) - f(X_{s-}^{u_0})) \right) = \mathbb{1}_{\{u_0 \in V_{s-}\}} \left(\mathbb{1}_{\{z \leq \lambda(\infty)\}} f(\infty) - f(\infty) \right) = 0$$

for any time s after the time when X^{u_0} reaches infinity. The reinfection events due to the reservoir thus do not contribute anymore to the dynamics when they concern a cell with an infinite quantity of parasites, even if λ is not bounded.

For the sake of readability we will assume that the processes under consideration in the sequel satisfy Assumption EU, and the conditions given in the cell division and death, but we will not indicate it.

We aim at investigating the long time behaviour of the infection in the cell population. As we have explained in the introduction, the strategy to obtain information at the population level is to introduce an auxiliary process behaving as a “typical individual”. A slight adaptation of Theorem 3.1 in [20] and Proposition 1 in [27] allows us to obtain the following Many-to-One formula (see ideas of the proof on page 31):

Proposition 1.2. *Let the process Z be defined as in Proposition 1.1, and recall notation (1.3). Then for $x_0 \geq 0$ and all measurable bounded functions $F : \mathbb{D}([0, t], \overline{\mathbb{R}}_+) \rightarrow \mathbb{R}$, we have:*

$$\mathbb{E}_{\delta_{x_0}} \left[\sum_{u \in V_t} F(X_s^u, s \leq t) \right] = e^{(b-d)t} \mathbb{E}_{x_0} [F(Y_s, s \leq t)], \quad (1.4)$$

where we recall that for $s \leq t$, X_s^u denotes the trait of the ancestor of individual $u \in V_t$ at time s . In (1.4), $Y = (Y_t : t \geq 0)$ is a time inhomogeneous diffusion with jumps, which is the unique

non-negative strong solution to

$$\begin{aligned}
 Y_t = x_0 &+ \int_0^t g(Y_s)ds + \int_0^t \sqrt{2\sigma^2(Y_s)}dB_s + \int_0^t \int_0^{p(Y_{s-})} \int_{\mathbb{R}_+} z\tilde{Q}(ds, dx, dz) \\
 &+ \int_0^t \int_0^{2b} \int_0^1 (\theta - 1)Y_{s-}N(ds, dx, d\theta) + \int_0^t \int_0^{r(\mathbb{E}_{x_0}[Y_s])} \int_{\mathbb{R}_+} \mathbf{p}N_1(ds, dx, d\mathbf{p}) \\
 &+ \int_0^t \int_0^{\lambda(Y_{s-})} \int_{\mathbb{R}_+} \mathbf{i}N_2(ds, dx, d\mathbf{i}), \tag{1.5}
 \end{aligned}$$

where $N(ds, dx, d\theta)$, $N_1(ds, dx, d\mathbf{p})$ and $N_2(ds, dx, d\mathbf{i})$ are independent Poisson Point Measures with intensities $ds \times dx \times \kappa(d\theta)$, $ds \times dx \times \mathfrak{P}(d\mathbf{p})$ and $ds \times dx \times \mathfrak{J}(d\mathbf{i})$, and Q and B have been defined in (1.1). N , N_1 , N_2 , Q and B are independent.

We notice that a ‘‘typical’’ lineage (i.e. following the auxiliary process) divides twice as fast as a lineage that would be followed over time ($2b$ instead of b). This bias is due to the fact that when a cell is chosen uniformly at random at a given time, one is more likely to choose a cell belonging to a lineage that has divided a lot. The doubling is due to the fact that one cell produces two daughter cells, and for a process with a random number of offspring at each reproduction, the bias corresponds to the average number of offspring (see [6] for more details).

We also notice that the death rate, d , is not present in the law of the auxiliary process. This is because the death rate of a cell does not depend on the amount of parasites it contains, and so cell deaths do not alter the law of the proportion of cells with a given amount of parasites (see the proof of Proposition 1.2 for more details).

Remark 1.3. Suppose that we take $r(Z_s)$ (same conclusion with $r(e^{-(b-d)s}\langle Z_s, \text{Id} \rangle)$) instead of $r(e^{-(b-d)s}\mathbb{E}_{\delta_{x_0}}[\langle Z_s, \text{Id} \rangle])$ in the SDE giving the process dynamics in Proposition 1.1, in order to take into account the quantity of parasites in the cell medium instead of its expectation. If we introduce the process ($W_t := e^{-(b-d)t}Z_t, t \geq 0$), Itô formula gives, for $f \in C_b^2(\mathbb{R}_+)$ and $t \geq 0$,

$$\begin{aligned}
 \frac{d\mathbb{E}[\langle W_t, f \rangle]}{dt} &= \mathbb{E}[\langle W_t, \mathcal{G}f \rangle] + 2b\mathbb{E}[\langle W_t, f(\Theta \cdot) - f(\cdot) \rangle] \\
 &+ \mathbb{E}[\langle W_t, \lambda(\cdot)(f(\cdot + \mathcal{I}) - f(\cdot)) \rangle] + \mathbb{E}[\langle W_t, r(W_t e^{(b-d)t})(f(\cdot + \mathcal{P}) - f(\cdot)) \rangle],
 \end{aligned}$$

where \mathcal{P} (resp. \mathcal{I}) denotes a random variable with law \mathfrak{P} (resp. \mathfrak{J}). The interaction appearing in the last term impedes finding an auxiliary process Y satisfying Proposition 1.2. This is why we made this simplification.

2. Results

We will now study how the interactions between the parasites growth rate, fluctuations and positive jumps, the rate and intensity of reinfections and the cell division rate and law shape the long time behaviour of the infection. To this aim, we define the function

$$\rho(x) := \mathbb{E}[\mathcal{I}]\lambda(x) + \mathbb{E}[\mathcal{P}]r(x) + g(x) - bx, \quad x \geq 0. \tag{2.1}$$

To know if the quantity of parasites can explode (reach ∞ in a finite time), we will contrast the two following assumptions:

(SN ∞) There exist $a < 1$ and a non-negative function f on \mathbb{R}_+ such that

$$\mathcal{D}(a, x) = -f(x) + o(\ln x), \quad (x \rightarrow +\infty).$$

(LN ∞) There exist $a \in \mathcal{A}$, $\eta > 0$ and $\bar{x} > 0$ such that for all $x \geq \bar{x}$

$$\mathcal{D}(a, x) \geq \ln x (\ln \ln x)^{1+\eta}.$$

Here,

$$\mathcal{A} := \{a > 1, \mathbb{E}[\Theta^{1-a}] < \infty\}, \quad (2.2)$$

and for $a \neq 1 \in \mathbb{R}_+^*$ and $x > 0$,

$$I_a(x) = ax^{-2} \int_{\mathbb{R}_+} z^2 \left(\int_0^1 (1 + zx^{-1}v)^{-1-a}(1-v)dv \right) \pi(dz) \quad (2.3)$$

and

$$\mathcal{D}(a, x) := \frac{g(x)}{x} - a \frac{\sigma^2(x)}{x^2} - p(x)I_a(x) + \lambda(x) \frac{\mathbb{E}[(1 + \mathcal{I}/x)^{1-a}] - 1}{1-a}. \quad (2.4)$$

Condition (SN ∞) ensures that Y does not explode, whereas under Condition (LN ∞), the process Y has a positive probability to reach ∞ in finite time (see [21] for the introduction of these conditions, being a refinement of conditions introduced in [19]).

We will assume in the sequel that $\mathbb{E}[\ln 1/\Theta] < \infty$.

2.1. Controlled or uncontrolled infection

First, we want to know whether the amount of parasites in the cells remains bounded.

Notice that under the assumptions of point (1) of the following proposition, Condition (SN ∞) holds. Depending on the relative strengths of the functions g, p, σ, λ and the parameter b , we may obtain contrasted long time behaviours for the infection in the cell population.

Proposition 2.1. *Let the process Z be defined as in Proposition 1.1, with $Z_0 = \delta_0$, and $N_t = \langle Z_t, 1 \rangle$ denote the number of cells alive at time $t \geq 0$.*

- (1) *If there exists \bar{x} such that for $x \geq \bar{x}$, $g(x) + \mathbb{E}[Z]\lambda(x) \leq \tilde{g}x$ with $\tilde{g} < b$, then there exists a constant C such that for any $K, t \geq 0$,*

$$\mathbb{E}_{\delta_0} \left[\mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^{(u)} \geq K\}}}{N_t} \right] \leq \frac{C}{\sqrt{K}}.$$

Notice that point (1) implies that for any $t \geq 0$

$$\lim_{K \rightarrow \infty} \mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^{(u)} \geq K\}}}{N_t} = 0, \quad \text{in probability.}$$

- (2) *If there exists $\tilde{g} > b$ such that $g(x) + \mathbb{E}[Z]\lambda(x) \geq \tilde{g}x$ for any $x > 0$, then for every $K \geq 0$,*

$$\lim_{t \rightarrow \infty} \mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{\sup_{s \leq t} X_s^{(u)} \leq K\}}}{N_t} = 0, \quad \text{in probability.}$$

- (3) *Under Condition (LN ∞) and if $\lambda(x) > 0$ for any $x \in \mathbb{R}_+^*$,*

$$\lim_{t \rightarrow \infty} \mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^{(u)} < \infty\}}}{N_t} = 0, \quad \text{a.s.}$$

Thus, in the first case, the proportion of cells containing a large amount of parasites is small, when considering any given time. In the second case, on the contrary, when the time is large, the lineage of each cell alive at time t has contained a high number of parasites at a given time with high probability. The third case is even more extreme as the quantity of parasites in most of the cells reaches infinity. One could imagine that the presence of a large number of parasites in a cell disturbs the functioning of this latter. The second and third cases could therefore correspond to the death of a large fraction of the cells for example if we allowed the cell death rate to depend on the quantity of parasites.

Interestingly, the parasite growth rate $g(\cdot)$ and the infection by the reservoir $\mathbb{E}[Z]\lambda(\cdot)$ have a similar role in the quantity of parasites on the long term (increasing any of them has the

same impact on the criteria of points (1) and (2)). This highlights the importance of taking into account reinfections by the reservoir (or similarly the immigration of infected individuals) when studying an epidemics.

2.2. The case without reservoir

We now wonder if the cell population is able to get rid of the infection in the case where it is not reinfected by new contacts with the parasites reservoir after the first infection of the population.

Proposition 2.2. *Let Z and N be defined as in Proposition 1.1. Assume that $\lambda(\cdot) \equiv 0$.*

(1) *Assume that g is a concave function and there exist $\alpha, \eta > 0$ such that*

$$\rho(x) \leq -(\alpha x \wedge \eta), \quad x \geq 0. \quad (2.5)$$

Then for every $\varepsilon > 0$ and $Z_0 = \delta_{x_0}$ with $x_0 < \infty$,

$$\lim_{t \rightarrow \infty} \mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^{(u)} > \varepsilon\}}}{N_t} = 0, \quad \text{in probability.}$$

(2) *Under Condition (LN ∞) and if $r(x_0) > 0$,*

$$\lim_{t \rightarrow \infty} \mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^{(u)} < \infty\}}}{N_t} = 0, \quad \text{a.s. when } Z_0 = \delta_{x_0}.$$

When $\lambda(\cdot) \equiv 0$, ρ writes

$$\rho(x) = \mathbb{E}[\mathcal{P}]r(x) + g(x) - bx, \quad x \geq 0.$$

Hence, when the cell division rate outweighs the growth of the parasites and the reinfections by parasites present in the medium, the quantity of parasites goes to 0 in a fraction of the cells close to 1. When, on the contrary, parameters of parasites growth and reinfection are larger than the cell division rate, the quantity of parasites goes to ∞ in a fraction of the cells close to 1.

The concavity of g seems a reasonable assumption. Indeed it means that there is a negative interaction between parasites, which may be due for instance to competition for space, or to the activation of the immune system. A logistic growth rate for example meets this assumption.

2.3. Coming down from infinity

We now assume that parasites present in the cell medium do not re infect other cells, for example because the immune system destroys them before they can attack other cells. However, we suppose that the cell population is frequently in contact with the parasite living in another environment (the reservoir). We investigate under which conditions on the frequency and rate of infection and on the dynamics of the parasites within each cell, the cell population is able to control the size of the parasites proliferation.

Recall the definition of the function $\mathcal{D}(a, \cdot)$ in (2.4). Then we have the following result.

Proposition 2.3. *Let the process Z be defined as in Proposition 1.1. Assume that $r(\cdot) \equiv 0$.*

(1) *If there exist $a \in \mathcal{A}$ and a non-negative function f such that*

$$\mathcal{D}(a, x) \geq f(x) + o(\ln x), \quad (x \rightarrow \infty), \quad (2.6)$$

then for every $K, t \geq 0$,

$$\lim_{x \rightarrow \infty} \mathbb{E}_{\delta_x} \left[\mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^{(u)} \leq K\}}}{N_t} \right] = 0.$$

(2) If there exist $0 < a < 1$ and $\eta > 0$, such that

$$\mathcal{D}(a, x) \leq -\ln x (\ln \ln x)^{1+\eta}, \quad (x \rightarrow \infty), \quad (2.7)$$

then for every $x, t \geq 0$,

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\delta_x} \left[\mathbf{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbf{1}_{\{X_t^{(u)} \geq K\}}}{N_t} \right] = 0.$$

There is thus an interplay between the increase of the number of parasites (by growth (g) or reinfection by the reservoir (λ)) and the fluctuations of the number of parasites (characterized by σ and p), which tend to decrease the number of parasites. This interplay is fully characterized by the class of functions $(\mathcal{D}(a, \cdot), a \neq 1)$.

If g and λ are large enough, the cell population is not able to get rid of the parasites when they are initially numerous (x large) and the quantity of parasites is large in a fraction of the cells close to 1. If on the contrary the noise, characterized by σ and p , outweighs the parasites growth and reinfection, the cell population succeeds in containing the infection.

3. Proofs

3.1. Proof of Proposition 1.2 and preliminary lemma

We first explain how the proofs of Proposition 1 in [27] and Theorem 3.1 in [20] must be modified to obtain Proposition 1.2.

Proof of Proposition 1.2. The existence and strong unicity of a solution to (1.5) are a consequence of Proposition 1 in [27], and accordingly, $(Y_t, t \geq 0)$ is a $[0, \infty]$ -valued process, which satisfies (1.5) up to the time $\tau_n := \inf\{t \geq 0 : Y_t \geq n\}$ for all $n \geq 1$, and $Y_t = \infty$ for all $t \geq \tau_\infty := \lim_{n \rightarrow \infty} \tau_n$. Notice that the process Y is not Markovian, as $r(\mathbb{E}[Y_t])$ is time dependent. In the statement of Proposition 1 in [27], the functions do not depend on time, unlike the case of our process. However this additional dependence does not bring any modification to the proofs (see [22, Appendix B] for more details).

The proof of the Many-to-One formula (1.4) is obtained by a slight modification of the proof of Theorem 3.1 in [20]. There are essentially three differences with the result of Theorem 3.1 in [20]. First a simplification: the division rate b does not depend on the quantity of parasites in the cells, which makes equations simpler. Second, the reinfection by parasites in the medium (resp. in the reservoir), which can be seen, at time t , as the death of a cell $u \in V_t$ with a quantity X_t^u of parasites and the birth of a cell with a quantity $X_t^u + \mathcal{I}$ of parasites, \mathcal{I} following the distribution \mathfrak{J} (resp. a quantity $X_t^u + \mathcal{P}$ of parasites, \mathcal{P} following the distribution \mathfrak{P}). The point is that, contrarily to the case considered in [20], this additional division rate is time inhomogeneous, and unlike the division rate b it may depend on the quantity of parasites. However it does not impact the mean number of cells, and the adaptation of the proof to this case is not difficult. The last difference is that we allow for the death of cells without producing any daughter cell. Here again the adaptation is fairly simple, because this death rate, d , does not depend on the quantity of parasites in the cell. To fix ideas, these modifications would lead to the following

equivalent of Equation (3.8) in [20]:

$$\begin{aligned}
 \mu_{t_0,s}(x_0, f) &= f(t_0, x_0) + \int_{t_0}^s \int_{\mathbb{R}_+} [\mathcal{G}f(r, x) - (b-d)f(r, x) + \partial_r f(r, x)] \mu_{t_0,r}(x_0, dx) dr \\
 &\quad + \int_{t_0}^s \int_{\mathbb{R}_+} \left[2b \int_0^1 f(r, \theta x) \kappa(d\theta) - (b+d)f(r, x) \right] \mu_{t_0,r}(x_0, dx) dr \\
 &\quad + \int_{t_0}^s \int_{\mathbb{R}_+} \left[r(\mathbb{E}[Y_r | Y_{t_0} = x_0]) \int_{\mathbb{R}_+} (f(r, x + \mathbf{p}) - f(r, x)) \mathfrak{P}(d\mathbf{p}) \right] \mu_{t_0,r}(x_0, dx) dr \\
 &\quad + \int_{t_0}^s \int_{\mathbb{R}_+} \left[\lambda(x) \int_{\mathbb{R}_+} (f(r, x + \mathbf{i}) - f(r, x)) \mathfrak{J}(d\mathbf{i}) \right] \mu_{t_0,r}(x_0, dx) dr.
 \end{aligned}$$

We see that the terms involving d cancel out. This explains why the death rate does not impact the dynamics of Y . Intuitively this is due to the fact that the death rate is constant. Hence all individuals die with the same probability and the behaviour of a “typical individual” is not modified. \square

We may now prove the results on the long time behaviour of the infection presented in Section 2.

Let us recall the definition of $(Y_t, t \geq 0)$ in (1.5), and introduce the process $(\bar{Y}_t, t \geq 0)$, solution to the SDE

$$\begin{aligned}
 \bar{Y}_t &= x_0 + \int_0^t g(\bar{Y}_s) ds + \int_0^t \sqrt{2\sigma^2(\bar{Y}_s)} dB_s + \int_0^t \int_0^{p(\bar{Y}_{s-})} \int_{\mathbb{R}_+} z \tilde{Q}(ds, dx, dz) \\
 &\quad + \int_0^t \int_0^{2b} \int_0^1 (\theta - 1) \bar{Y}_{s-} N(ds, dx, d\theta) \\
 &\quad + \int_0^t \int_0^{r(\bar{y})} \int_{\mathbb{R}_+} \mathbf{p} N_1(ds, dx, d\mathbf{p}) + \int_0^t \int_0^{\lambda(\bar{Y}_{s-})} \int_{\mathbb{R}_+} \mathbf{i} N_2(ds, dx, d\mathbf{i}), \quad (3.1)
 \end{aligned}$$

where the Brownian B and the Poisson random measures Q , N , N_1 and N_2 are the same as in the SDE (1.5), and \bar{y} is a non-negative real number, which will be chosen later.

Moreover, for the sake of readability we introduce the following notations:

- For $x > 0$,

$$\tau^+(x) := \inf\{t \geq 0, Y_t \geq x\} \quad \text{and} \quad \tau^-(x) := \inf\{t \geq 0, Y_t \leq x\}. \quad (3.2)$$

- For $x > 0$,

$$\bar{\tau}^+(x) := \inf\{t \geq 0, \bar{Y}_t \geq x\} \quad \text{and} \quad \bar{\tau}^-(x) := \inf\{t \geq 0, \bar{Y}_t \leq x\}.$$

-

$$\tau^+(\infty) = \lim_{x \rightarrow \infty} \tau^+(x) \quad \text{and} \quad \bar{\tau}^+(\infty) = \lim_{x \rightarrow \infty} \bar{\tau}^+(x). \quad (3.3)$$

Before giving the main proofs, we will derive a lemma, which will be useful on several occasions. It concerns the probability for the processes Y and \bar{Y} to reach infinity in a finite time.

Lemma 3.1.

- (i) *If Condition (SN ∞) holds, then*

$$\mathbb{P}_x(\tau^+(\infty) < \infty) = \mathbb{P}_x(\bar{\tau}^+(\infty) < \infty) = 0 \quad \text{for all } x > 0.$$

- (ii) *If Condition (LN ∞) holds then $\mathbb{P}_x(\tau^+(\infty) < \infty) > 0$ and $\mathbb{P}_x(\bar{\tau}^+(\infty) < \infty) > 0$ for all large enough $x > 0$.*

Proof. Let us recall the definition of \mathcal{A} in (2.2) and introduce the set of functions $G_a^{(t)}$ given for $a \in \mathcal{A} \cup (0, 1)$, $t \geq 0$ and $x_0 \geq 0, x > 0$ by

$$G_a^{(t)}(x_0, x) := (a - 1) \left(\frac{g(x)}{x} - a \frac{\sigma^2(x)}{x^2} - 2b \frac{1 - \mathbb{E}[\Theta^{1-a}]}{1 - a} - p(x)I_a(x) \right. \\ \left. + \lambda(x) \mathbb{E} \left[\frac{(1 + \mathcal{I}/x)^{1-a} - 1}{1 - a} \right] \right) - r(\mathbb{E}_{x_0}[Y_t]) \mathbb{E} \left[\left(1 + \frac{\mathcal{P}}{x} \right)^{1-a} - 1 \right], \quad (3.4)$$

where I_a has been defined in (2.3). For all $h > c > 0$, let $T = \tau^-(c) \wedge \tau^+(h)$. Then, for all $a \in \mathcal{A} \cup (0, 1)$, the process

$$Z_{t \wedge T}^{(a)} := (Y_{t \wedge T})^{1-a} \exp \left(\int_0^{t \wedge T} G_a^{(s)}(x_0, Y_s) ds \right) \quad (3.5)$$

is a \mathcal{F}_t -martingale. The proof of this property is the same as the proof of [19, Lemma 5.1] or [21, Lemma 7.1].

Now we see that Condition (LN ∞) is equivalent to the existence of $a \in \mathcal{A}$ and $\eta > 0$ such that for all $t \geq 0$

$$G_a^{(t)}(x_0, x) \geq \ln x (\ln(\ln x))^{1+\eta}, \quad (x \rightarrow +\infty),$$

and that Condition (SN ∞) is equivalent to the existence of $a < 1$ and a non-negative function f on \mathbb{R}_+ such that for all $t \geq 0$,

$$G_a^{(t)}(x_0, x) \geq f(x) + o(\ln x), \quad (x \rightarrow +\infty).$$

Indeed, the second to last term in (3.4) has no influence as r is a bounded function: this term is thus bounded on \mathbb{R}_+ .

Introducing similarly the set of functions \bar{G}_a , for $a \in \mathcal{A} \cup (0, 1)$, $x_0 \leq \bar{y}$ and $x > 0$ via

$$\bar{G}_a(x) := (a - 1) \left(\frac{g(x)}{x} - a \frac{\sigma^2(x)}{x^2} - 2b \frac{1 - \mathbb{E}[\Theta^{1-a}]}{1 - a} - p(x)I_a(x) + \lambda(x) \mathbb{E} \left[\frac{(1 + \mathcal{I}/x)^{1-a} - 1}{1 - a} \right] \right) \\ - r(\bar{y}) \mathbb{E} \left[\left(1 + \frac{\mathcal{P}}{x} \right)^{1-a} - 1 \right],$$

we obtain that for all $a \in \mathcal{A} \cup (0, 1)$, the process

$$\bar{Z}_{t \wedge T}^{(a)} := (\bar{Y}_{t \wedge T})^{1-a} \exp \left(\int_0^{t \wedge T} \bar{G}_a(Y_s) ds \right)$$

is a \mathcal{F}_t -martingale, and we have the same equivalences with $\bar{G}_a(\cdot)$ in place of $G_a^{(t)}(x_0, \cdot)$.

The proof of Lemma 3.1 is therefore a direct application of points i) and ii) of [21, Theorem 4.1]. \square

The proof strategies are similar for the three main results of this paper. We first derive a property on the auxiliary process and then show how we can infer the long time behaviour of the infection at the cell population scale.

3.2. Proof of Proposition 2.1

We derive an auxiliary Lemma before proving the proposition. Recall the definition of ρ in (2.1).

Lemma 3.2. *If there exists \bar{x} such that for $x \geq \bar{x}$, $g(x) + \mathbb{E}[T]\lambda(x) \leq \tilde{g}x$ with $\tilde{g} < b$, then there exist $y \geq \bar{x}$ and $M, \varepsilon > 0$ such that for every $x \geq \bar{y}$,*

$$\rho(x) \leq -M - (b - \tilde{g})\varepsilon,$$

and

$$\mathbb{E}[Y_t] \leq \bar{y}, \quad \text{for all } t \geq 0.$$

Proof. We first make the simplifying assumption that

$$g(x) + \mathbb{E}[\mathcal{I}]\lambda(x) = \tilde{g}x, \quad x \geq \bar{x}.$$

In particular, for $x \geq \bar{x}$

$$\rho(x) = (\tilde{g} - b)x + \mathbb{E}[\mathcal{P}]r(x), \quad (3.6)$$

which goes to $-\infty$ when x goes to infinity, as $\tilde{g} < b$. This implies the existence of $\bar{y} \geq \bar{x}$ and $\varepsilon > 0$ such that for any $x \geq \bar{y}$,

$$\rho(x) \leq -M - (b - \tilde{g})\varepsilon, \quad (3.7)$$

where

$$M := \sup_{x \leq \bar{x}} \{g(x) + \mathbb{E}[\mathcal{I}]\lambda(x)\}.$$

We choose such a real number \bar{y} in the definition of \bar{Y} in (3.1).

The proof of Lemma 3.2 will stem from the following two properties:

- (a) for every $t \geq 0$, $\mathbb{E}[\bar{Y}_t] \leq \bar{y}$,
- (b) for every $t \geq 0$, $Y_t \leq \bar{Y}_t$ a.s.

Recall (3.1). Applying Itô formula with jumps (see for instance [16] Th 5.1 p. 66) and using (3.6), we get

$$\begin{aligned} e^{(b-\tilde{g})t}\bar{Y}_t &= \int_0^t e^{(b-\tilde{g})s} \sqrt{2\sigma^2(\bar{Y}_s)} dB_s + \int_0^t \int_0^{p(\bar{Y}_{s-})} \int_{\mathbb{R}_+} e^{(b-\tilde{g})s} z \tilde{Q}(ds, dx, dz) \\ &\quad + \int_0^t \int_0^{2b} \int_0^1 e^{(b-\tilde{g})s} (\theta - 1) \bar{Y}_{s-} \tilde{N}(ds, dx, d\theta) \\ &\quad + \int_0^t e^{(b-\tilde{g})s} (g(\bar{Y}_s) + \mathbb{E}[\mathcal{I}]\lambda(\bar{Y}_s) - \tilde{g}\bar{Y}_s) \mathbb{1}_{\{\bar{Y}_s \leq \bar{x}\}} ds \\ &\quad + \int_0^t \int_0^{r(\bar{y})} \int_{\mathbb{R}_+} e^{(b-\tilde{g})s} \mathbf{p} N_1(ds, dx, d\mathbf{p}) + \int_0^t \int_0^{\lambda(\bar{Y}_{s-})} \int_{\mathbb{R}_+} e^{(b-\tilde{g})s} \mathbf{i} \tilde{N}_2(ds, dx, d\mathbf{i}) \\ &= M_t^{(loc)} + \int_0^t \int_0^{r(\bar{y})} \int_{\mathbb{R}_+} e^{(b-\tilde{g})s} \mathbf{p} N_1(ds, dx, d\mathbf{p}) \\ &\quad + \int_0^t e^{(b-\tilde{g})s} (g(\bar{Y}_s) + \mathbb{E}[\mathcal{I}]\lambda(\bar{Y}_s) - \tilde{g}\bar{Y}_s) \mathbb{1}_{\{\bar{Y}_s \leq \bar{x}\}} ds, \end{aligned}$$

where \tilde{N} and \tilde{N}_2 are the compensated measures of N and N_2 , and $(M_{t \wedge \tau^+(x)}^{(loc)}, t \geq 0)$ is a martingale for any $x > 0$. Indeed this is a local martingale and we can check using [28, Theorem 51 p. 38] as in the proof of Lemma 5.1 in [19] that it is a martingale. Taking the expectation and using (3.7) and that $b > \tilde{g}$ yield

$$\mathbb{E}[e^{(b-\tilde{g})(t \wedge \tau^+(x))} \bar{Y}_{t \wedge \tau^+(x)}] \leq (r(\bar{y})\mathbb{E}[\mathcal{P}] + M) \mathbb{E} \left[\frac{e^{(b-\tilde{g})(t \wedge \tau^+(x))}}{b - \tilde{g}} \right]. \quad (3.8)$$

As $g(x) + \mathbb{E}[\mathcal{I}]\lambda(x) = \tilde{g}x$ for large x , condition (SN_∞) is satisfied and thus according to Lemma 3.1,

$$\lim_{x \rightarrow \infty} \bar{\tau}^+(x) = \infty.$$

As a consequence, letting x go to infinity in (3.8), we obtain by Fatou's Lemma:

$$\begin{aligned} e^{(b-\tilde{g})t}\mathbb{E}[\bar{Y}_t] &= \mathbb{E}\left[\liminf_{x\rightarrow\infty} e^{(b-\tilde{g})(t\wedge\tau^+(x))}\bar{Y}_{t\wedge\bar{\tau}^+(x)}\right] \\ &\leq \liminf_{x\rightarrow\infty} \mathbb{E}\left[e^{(b-\tilde{g})(t\wedge\tau^+(x))}\bar{Y}_{t\wedge\bar{\tau}^+(x)}\right] \\ &\leq (r(\bar{y})\mathbb{E}[\mathcal{P}] + M) \frac{e^{(b-\tilde{g})t}}{b-\tilde{g}}, \end{aligned}$$

and thus

$$\mathbb{E}[\bar{Y}_t] \leq \frac{r(\bar{y})\mathbb{E}[\mathcal{P}] + M}{b-\tilde{g}} \leq \bar{y} - \varepsilon.$$

This concludes the proof of step (a).

Let us now move on to step (b). By unicity of the solutions to (1.5) and (3.1) and as the rate of positive jumps p is non decreasing with the process value, we have that for $t \geq 0$:

$$Y_t \leq \bar{Y}_t \quad \text{a.s. if } \mathbb{E}[Y_s] \leq \bar{y} \quad \forall s \leq t.$$

We will show that this last inequality is satisfied for every $s \geq 0$.

Notice first that with computations similar to the ones to get (3.8), we obtain:

$$\begin{aligned} \mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}] &= \mathbb{E}[\mathcal{P}] \int_0^t \mathbb{E}\left[\mathbf{1}_{s\leq\tau^+(x)} e^{(b-\tilde{g})s} r(\mathbb{E}[Y_s])\right] ds \\ &\quad + \int_0^t \mathbb{E}\left[\mathbf{1}_{s\leq\tau^+(x)} e^{(b-\tilde{g})s} (g(Y_s) + \mathbb{E}[\mathcal{I}]\lambda(Y_s) - \tilde{g}Y_s)\mathbf{1}_{Y_s\leq\bar{x}}\right] ds. \end{aligned} \quad (3.9)$$

Hence for a given x , $\mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}]$ evolves continuously with t . This justifies the existence of $\mu(x)$ and $\nu(x)$ defined as follows: assume that there exist $(x, t) \in \mathbb{R}_+^2$ such that

$$\mathbb{E}\left[e^{(b-\tilde{g})(t\wedge\tau^+(x))}\bar{Y}_{t\wedge\bar{\tau}^+(x)}\right] e^{-(b-\tilde{g})t} = \bar{y} + \varepsilon,$$

and choose the smallest t realising this equality:

$$\mu(x) := \inf\left\{t \geq 0, \mathbb{E}\left[e^{(b-\tilde{g})(t\wedge\tau^+(x))}\bar{Y}_{t\wedge\bar{\tau}^+(x)}\right] e^{-(b-\tilde{g})t} = \bar{y} + \varepsilon\right\}$$

(if $\mu(x) = \infty$, we may jump directly to Equation (3.10)). Now, define $\nu(x) \geq \mu(x)$ as follows:

$$\nu(x) := \inf\left\{t \geq \mu(x), \mathbb{E}\left[e^{(b-\tilde{g})(t\wedge\tau^+(x))}\bar{Y}_{t\wedge\bar{\tau}^+(x)}\right] e^{-(b-\tilde{g})t} \notin [\bar{y}, \bar{y} + 2\varepsilon]\right\},$$

where $\varepsilon > 0$.

Notice that (3.9) may be rewritten, for $\mu(x) \leq t \leq \nu(x)$,

$$\begin{aligned} &\mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}] - \mathbb{E}[e^{(b-\tilde{g})(\mu(x)\wedge\tau^+(x))}Y_{\sigma\wedge\tau^+(x)}] \\ &= \int_{\mu(x)}^t \mathbb{E}\left[\mathbf{1}_{s\leq\tau^+(x)} e^{(b-\tilde{g})s} \left(\mathbb{E}[\mathcal{P}]r(\mathbb{E}[e^{(b-\tilde{g})(s\wedge\tau^+(x))}Y_{s\wedge\tau^+(x)}])e^{-(b-\tilde{g})s}\right.\right. \\ &\quad \left.\left.+ (g(Y_s) + \mathbb{E}[\mathcal{I}]\lambda(Y_s) - \tilde{g}Y_s)\mathbf{1}_{Y_s\leq\bar{x}}\right)\right] ds \\ &= \int_{\mu(x)}^t \mathbb{E}\left[\mathbf{1}_{s\leq\tau^+(x)} e^{(b-\tilde{g})(t\wedge\tau^+(x))s} \left(\rho(\mathbb{E}[e^{(b-\tilde{g})(s\wedge\tau^+(x))}Y_{s\wedge\tau^+(x)}])e^{-(b-\tilde{g})s}\right.\right. \\ &\quad \left.\left.+ (b-\tilde{g})\mathbb{E}[e^{(b-\tilde{g})(s\wedge\tau^+(x))}Y_{s\wedge\tau^+(x)}]e^{-(b-\tilde{g})s} + (g(Y_s) + \mathbb{E}[\mathcal{I}]\lambda(Y_s) - \tilde{g}Y_s)\mathbf{1}_{Y_s\leq\bar{x}}\right)\right] ds. \end{aligned}$$

Using (3.7) and the fact that $\bar{y} \geq \bar{x}$, we obtain:

$$\begin{aligned} \mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}] &\leq \mathbb{E}[e^{(b-\tilde{g})(\mu(x)\wedge\tau^+(x))}Y_{\sigma\wedge\tau^+(x)}] \\ &\quad + (b-\tilde{g})(t\wedge\tau^+(x)) \int_{\mu(x)}^t \mathbb{E}\left[\mathbf{1}_{s\leq\tau^+(x)} \mathbb{E}[e^{(b-\tilde{g})(s\wedge\tau^+(x))}Y_{s\wedge\tau^+(x)}]\right] ds. \end{aligned}$$

Applying Grönwall's Lemma then yields

$$\mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}] \leq \mathbb{E}[e^{(b-\tilde{g})(\mu(x)\wedge\tau^+(x))}Y_{\mu(x)\wedge\tau^+(x)}]e^{(b-\tilde{g})(t-\mu(x))},$$

or equivalently

$$\mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}]e^{-(b-\tilde{g})t} \leq \mathbb{E}[e^{(b-\tilde{g})(\mu(x)\wedge\tau^+(x))}Y_{\mu(x)\wedge\tau^+(x)}]e^{-(b-\tilde{g})\mu(x)} = \bar{y} + \varepsilon.$$

This implies that for any $(t, x) \in \mathbb{R}_+^2$,

$$\mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}]e^{-(b-\tilde{g})t} \leq \bar{y} + \varepsilon, \quad (3.10)$$

and thus for any $t \geq 0$, by Fatou's Lemma, as Condition (SN ∞) is satisfied

$$\begin{aligned} \mathbb{E}[Y_t] &= \mathbb{E}[e^{(b-\tilde{g})t}Y_t]e^{-(b-\tilde{g})t} = \mathbb{E}[\liminf_{x \rightarrow \infty} e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}]e^{-(b-\tilde{g})t} \\ &\leq \liminf_{x \rightarrow \infty} \mathbb{E}[e^{(b-\tilde{g})(t\wedge\tau^+(x))}Y_{t\wedge\tau^+(x)}]e^{-(b-\tilde{g})t} \leq \bar{y} + \varepsilon. \end{aligned}$$

This concludes step (b) in the case $g(x) + \mathbb{E}[\mathcal{I}]\lambda(x) = \tilde{g}x$ for $x \geq \bar{x}$. Now, as p and r are non decreasing, if we choose the same Poisson Point processes N , Q , N_1 and N_2 , and Brownian motion B we may couple two solutions $Y^{(1)}$ and $Y^{(2)}$ of the SDE (1.5) with respective growth rate functions for the parasites g_1 and g_2 satisfying

$$g_1(x) \leq g_2(x) \quad \text{for all } x \geq 0$$

and respective reservoir infection rates λ_1 and λ_2 satisfying

$$\lambda_1(x) \leq \lambda_2(x) \quad \text{for all } x \geq 0$$

such that

$$Y_t^{(1)} \leq Y_t^{(2)} \quad a.s. \quad \forall t \geq 0$$

as soon as $Y_0^{(1)} = Y_0^{(2)}$. The choices $g_1(x) = g(x)$, $\lambda_1(x) = \lambda(x)$ and g_2 and λ_2 continuous satisfying

$$g_2(x) + \mathbb{E}[\mathcal{I}]\lambda_2(x) \begin{cases} \geq g(x) + \mathbb{E}[\mathcal{I}]\lambda(x) & \text{for } x \leq \bar{x} \\ = \tilde{g}x & \text{for } x \geq \bar{x}, \end{cases}$$

allow us to extend the proof of step (b) to the assumptions of Lemma 3.2. \square

Proof of Proposition 2.1. Let us begin with point (1). Recall that from Lemma 3.2 $\mathbb{E}[Y_t] \leq \bar{y}$ for every $t \geq 0$. In particular, from Markov's inequality we have for every $K > 0$,

$$\mathbb{P}(Y_t \geq K) \leq \frac{\mathbb{E}[Y_t]}{K} \leq \frac{\bar{y}}{K}.$$

And from the Many-to-One formula (1.4) applied to the bounded function $x \mapsto \mathbf{1}_{\{x \geq K\}}$, we deduce

$$e^{-(b-d)t} \mathbb{E} \left[\mathbf{1}_{\{N_t \geq 1\}} \sum_{u \in V_t} \mathbf{1}_{\{X_t^u \geq K\}} \right] \leq \frac{\bar{y}}{K}. \quad (3.11)$$

According to Equations (7)–(8) in [13], if we denote by α_t and β_t the following variables,

$$\alpha_t := \frac{de^{(b-d)t} - d}{be^{(b-d)t} - d} \quad \text{and} \quad \beta_t := \frac{be^{(b-d)t} - b}{be^{(b-d)t} - d},$$

we have for $t \geq 0$,

$$\mathbb{P}(N_t = 0) = \alpha_t \quad \text{and for } n \geq 1, \quad \mathbb{P}(N_t = n) = (1 - \alpha_t)(1 - \beta_t)\beta_t^{n-1}.$$

We thus obtain, for $t \geq 0$ and $K \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}\left(N_t e^{-(b-d)t} \leq \frac{1}{\sqrt{K}}, N_t \geq 1\right) &= \sum_{i=1}^{\lfloor e^{(b-d)t/\sqrt{K}} \rfloor} \mathbb{P}(N_t = i) \\ &= (1 - \alpha_t) \left(1 - \beta_t^{\lfloor e^{(b-d)t/\sqrt{K}} \rfloor}\right) \\ &= \left(1 - \frac{d}{b} + O(e^{-(b-d)t})\right) \left(\frac{b-d}{b\sqrt{K}} + O\left(\frac{e^{-(b-d)t}}{\sqrt{K}}\right)\right) \\ &= \frac{1}{\sqrt{K}} \left(\frac{b-d}{b}\right)^2 + O\left(\frac{e^{-(b-d)t}}{\sqrt{K}}\right). \end{aligned}$$

We thus get, using (3.11),

$$\begin{aligned} \mathbb{E}\left[\mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^u \geq K\}}}{N_t}\right] &= \mathbb{E}\left[\frac{\sum_{u \in V_t} \mathbb{1}_{\{X_t^u \geq K\}}}{e^{(b-d)t}} \frac{\mathbb{1}_{\{N_t \geq 1\}}}{N_t e^{-(b-d)t}}\right] \\ &\leq \mathbb{P}\left(N_t e^{-(b-d)t} \leq \frac{1}{\sqrt{K}}, N_t \geq 1\right) + \frac{\bar{y}}{K} \sqrt{K} \\ &= \frac{1}{\sqrt{K}} \left(\left(\frac{b-d}{b}\right)^2 + \bar{y}\right) + O\left(\frac{e^{-(b-d)t}}{\sqrt{K}}\right), \end{aligned}$$

which ends the proof of point (1).

Let us now prove point (2). First, from the Many-to-One formula (1.4), we know that for $t, K \geq 0$,

$$\mathbb{E}\left[\mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{\sup_{s \leq t} X_s^{(u)} \leq K\}}}{e^{(b-d)t}}\right] = \mathbb{P}\left(\sup_{s \leq t} Y_s \leq K\right). \quad (3.12)$$

We will thus prove that the right hand term goes to 0 as t goes to infinity. Again we make simplifying assumptions and will obtain the general case by coupling. We thus assume that $g(x) + \mathbb{E}[\mathcal{I}]\lambda(x) = \tilde{g}x$ with $\tilde{g} > b$. In this case, from similar computations to the ones to get (3.8), we derive the following series of inequalities for $0 \leq y \leq x$,

$$\mathbb{E}_y[e^{(b-\tilde{g})(t \wedge \tau^+(x))} Y_{t \wedge \tau^+(x)}] = y + \mathbb{E}[\mathcal{P}] \int_0^t \mathbb{E}_y\left[\mathbb{1}_{s \leq \tau^+(x)} e^{(b-\tilde{g})s} r(\mathbb{E}_y[Y_s])\right] ds \geq y. \quad (3.13)$$

As $g(x) + \mathbb{E}[\mathcal{I}]\lambda(x) = \tilde{g}x$, condition (SN ∞) is satisfied and thus according to Lemma 3.1,

$$\limsup_{x \rightarrow \infty} \tau^+(x) = \infty, \quad \mathbb{P}_y\text{-a.s.}$$

We will now prove that (3.13) implies that for all $x \geq 0$,

$$\mathbb{P}_y(\tau^+(x) = \infty) < 1. \quad (3.14)$$

To this aim we will make a reductio ad absurdum by assuming that there exists $\bar{x} \geq 0$ such that

$$\mathbb{P}_y(\tau^+(\bar{x}) = \infty) = 1.$$

Notice that \bar{x} may be chosen larger than y without loss of generality. We thus have, for any $t \geq 0$,

$$Y_{t \wedge \tau^+(\bar{x})} \leq \bar{x} \quad \text{and} \quad e^{(b-\tilde{g})(t \wedge \tau^+(\bar{x}))} = e^{(b-\tilde{g})t}, \quad a.s.$$

Hence, taking $x = \bar{x}$ in (3.13) we obtain that for any $t \geq 0$,

$$\bar{x} e^{(b-\tilde{g})t} \geq y.$$

But by assumption, the left hand term goes to 0 when t goes to infinity, which leads to a contradiction. We thus conclude that (3.14) holds.

As a consequence, for every $x \geq 0$, there exist $\alpha_x > 0$ and $t_x < \infty$ such that

$$\mathbb{P}(\tau^+(x) \leq t_x) \geq \alpha_x.$$

Notice that as the functions p and r are non-decreasing, the solution of (1.5) is non decreasing with the initial condition. In particular, for any $y \geq 0$,

$$\mathbb{P}(\tau^+(x) > t_x) = \mathbb{P}_0(\tau^+(x) > t_x) \geq \mathbb{P}_y(\tau^+(x) > t_x).$$

Thus for any $n \in \mathbb{N}^*$,

$$\begin{aligned} \mathbb{P}(\tau^+(x) > nt_x) &= \mathbb{P}\left(\tau^+(x) > (n-1)t_x\right) \mathbb{P}\left(\tau^+(x) > nt_x \mid \tau^+(x) > (n-1)t_x\right) \\ &\leq \mathbb{P}\left(\tau^+(x) > (n-1)t_x\right) \sup_{y \geq 0} \mathbb{P}_y\left(\tau^+(x) > t_x\right) \\ &= \mathbb{P}\left(\tau^+(x) > (n-1)t_x\right) \mathbb{P}_0\left(\tau^+(x) > t_x\right) \\ &\leq \mathbb{P}\left(\tau^+(x) > (n-1)t_x\right) (1 - \alpha_x). \end{aligned}$$

By recurrence,

$$\mathbb{P}(\tau^+(x) > nt_x) \leq (1 - \alpha_x)^n,$$

and taking the limit when n goes to infinity, we obtain that

$$\mathbb{P}(\tau^+(x) < \infty) = 1 - \mathbb{P}(\tau^+(x) = \infty) = 1.$$

In particular, for any $K, \varepsilon > 0$, there exists $t_K < \infty$ such that for every $t \geq t_K$

$$\mathbb{P}\left(\sup_{s \leq t} Y_s \leq K\right) \leq \mathbb{P}\left(\sup_{s \leq t_K} Y_s \leq K\right) = \mathbb{P}\left(\tau^+(K) \geq t_K\right) \leq \varepsilon. \quad (3.15)$$

We thus deduce, from (3.12) and (3.15), that

$$\frac{\sum_{u \in V_t} \mathbb{1}_{\{\sup_{s \leq t} X_s^{(u)} \leq K\}}}{e^{(b-d)t}} \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in probability.}$$

Moreover, the fact that $(N_t, t \geq 0)$ is a birth and death process with individual death rate d and individual birth rate b also entails that $N_t e^{-(b-d)t}$ converges in probability to an exponential random variable with parameter $b - d$ on the event of survival, when t goes to infinity (see [13, Equation (15)] for instance). Hence, we have

$$\mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{\sup_{s \leq t} X_s^{(u)} \leq K\}}}{N_t} = \frac{\sum_{u \in V_t} \mathbb{1}_{\{\sup_{s \leq t} X_s^{(u)} \leq K\}}}{e^{(b-d)t}} \times \frac{\mathbb{1}_{\{N_t \geq 1\}}}{N_t e^{-(b-d)t}} \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in probability.}$$

To end the proof of point (2) we use again the fact that we may couple a solution $Y^{(1)}$ of (1.5) with $g_1(x) + \mathbb{E}[Z]\lambda_1(x) = \tilde{g}x$ with a solution $Y^{(2)}$ with $g_2(x) + \mathbb{E}[Z]\lambda_2(x) \geq \tilde{g}x$ with the same initial condition in such a way that

$$Y_t^{(1)} \leq Y_t^{(2)} \quad \forall t \geq 0 \quad a.s.$$

Let us end with the proof of point (3). The proof strategy is the following: we will introduce two processes \tilde{Y} and $\tilde{\tilde{Y}}$ such that for an equal initial condition of the processes $(Y, \tilde{Y}, \tilde{\tilde{Y}})$, there exists a coupling such that

$$\tilde{Y}_t \leq \tilde{\tilde{Y}}_t \leq Y_t, \quad a.s. \quad \forall t \geq 0.$$

We will furthermore show the existence of $C(\mu_0), \alpha > 0, x_1, T_1, \mu_0 < \infty$ such that (3.17) and (3.19) hold. We will then deduce the needed result.

The process \tilde{Y} defined as the unique strong solution to the following SDE

$$\begin{aligned} \tilde{Y}_t = & \int_0^t g(\tilde{Y}_s) ds + \int_0^t \sqrt{2\sigma^2(\tilde{Y}_s)} dB_s + \int_0^t \int_0^{p(\tilde{Y}_{s-})} \int_{\mathbb{R}_+} z \tilde{Q}(ds, dx, dz) \\ & + \int_0^t \int_0^{2b} \int_{\mathbb{R}_+} (\theta - 1) \tilde{Y}_{s-} N(ds, dx, d\theta), \end{aligned} \quad (3.16)$$

where the Brownian B and the Poisson random measures N and Q are the same as in the SDE (1.5) solved by the process $(Y_t, t \geq 0)$. Following the same proof as the one of Lemma 3.1 (it is enough to take $r \equiv 0$), we may check that Lemma 3.1 also holds for the process \tilde{Y} . Moreover, a close look at the proof of [19, Theorem 2.8] shows that the probability for \tilde{Y} to reach infinity in a given time is uniformly lower bounded for an initial condition large enough in the following sense:

$$\exists \alpha > 0, x_1, T_1 < \infty, \quad \inf_{x \geq x_1} \mathbb{P}_x(\tilde{\tau}^+(\infty) < T_1) \geq \alpha, \quad (3.17)$$

where $\tilde{\tau}^+(\infty)$ is defined as $\tau^+(\infty)$ in (3.3) with the process \tilde{Y} instead of Y . To be more precise, it is due to the fact that if we introduce the function

$$t(x) := \left(\ln(\ln(x^{(1-\delta)})) \right)^{-1-\eta},$$

where $\delta \leq (3 - 2a)^{-1}$, and the stopping times

$$\tilde{\tau}_0 = 0 \quad \text{and} \quad \tilde{\tau}_{n+1} := \tilde{\tau}^+ \left(\left(\tilde{Y}_{\tilde{\tau}_n} \vee 1 \right)^{1+\delta} \right) + \tilde{\tau}_n \quad \text{for } n \in \mathbb{N},$$

then under $\mathbb{P}_{\varepsilon^{-1}}$, if $\tilde{\tau}_n < \infty$ for all $n \geq 1$,

$$\sum_{n=1}^{\infty} t(\tilde{Y}_{\tilde{\tau}_n}) \leq (n \ln(1 + \delta))^{-(1+\delta)} =: \mathfrak{B} < \infty,$$

and for small enough ε ,

$$\mathbb{P}_{\varepsilon^{-1}} \left(\tilde{\tau}^+(\infty) < \mathfrak{B} \right) \geq \prod_{n=1}^{\infty} \left(1 - \varepsilon^{(a-1)\delta(1+\delta)^n} \right) > 0.$$

As the last quantity is decreasing with ε , we get (3.17).

Let us now introduce the process $\tilde{\tilde{Y}}$ defined as the unique strong solution to the following SDE

$$\begin{aligned} \tilde{\tilde{Y}}_t = & \int_0^t g(\tilde{\tilde{Y}}_s) ds + \int_0^t \sqrt{2\sigma^2(\tilde{\tilde{Y}}_s)} dB_s + \int_0^t \int_0^{p(\tilde{\tilde{Y}}_{s-})} \int_{\mathbb{R}_+} z \tilde{Q}(ds, dx, dz) \\ & + \int_0^t \int_0^{2b} \int_0^1 (\theta - 1) \tilde{\tilde{Y}}_{s-} N(ds, dx, d\theta) + \int_0^t \int_0^{\underline{\lambda}} \int_{\mathbb{R}_+} i N_2(ds, dx, di), \end{aligned} \quad (3.18)$$

where $\underline{\lambda} := \min_{x \leq 3x_1} \lambda(x)$, the Brownian B and the Poisson random measures N , Q and N_2 are the same as in the SDE (1.5) solved by the process $(Y_t, t \geq 0)$. Now denote:

$$A(t) := \int_0^t \sqrt{2\sigma^2(\tilde{\tilde{Y}}_s)} dB_s + \int_0^t \int_0^{p(\tilde{\tilde{Y}}_{s-})} \int_{\mathbb{R}_+} z \tilde{Q}(ds, dx, dz) + \int_0^t \int_0^{2b} \int_0^1 (\theta - 1) \tilde{\tilde{Y}}_{s-} N(ds, dx, d\theta)$$

and

$$B(t) := \int_0^t \int_0^{\underline{\lambda}} \int_{\mathbb{R}_+} i N_2(ds, dx, di).$$

If we take $\mu > 0$, introduce $\tilde{\tau}^+$ as τ^+ with \tilde{Y} in place of Y (recall Equation (3.2)), and make similar computations to the ones to prove [19, Lemma 5.1], followed by the Markov inequality, we get

$$\mathbb{P}_0 \left(\sup_{t \leq \mu} |A(t \wedge \tilde{\tau}^+(3x_1))| > \frac{x_1}{2} \right) \leq C_1 \mu \left(\sup_{x \leq 3x_1} \sigma(x) + \sup_{x \leq 3x_1} p(x) + 1 \right),$$

where C_1 is a finite constant. Moreover, as $B(t)$ is a compound Poisson process, with rate $0 < \underline{\lambda} := \min_{x \leq 3x_1} \lambda(x) < \infty$, there exists a positive constant C_2 depending on μ such that

$$\mathbb{P}_0 \left(\frac{3}{2}x_1 \leq B(\mu) \leq 2x_1 \right) \geq C_2(\mu).$$

If such a constant does not exist for a given x_1 because the probability \mathfrak{J} has his support on $(3/2x_1, \infty)$ for instance, we can choose a larger x_1 and all the properties derived until now still hold true.

As a consequence, for x_1 large enough, we have the following series of inequalities:

$$\begin{aligned} \mathbb{P}_0(\tilde{\tau}^+(x_1) \leq \mu) &\geq \mathbb{P}_0 \left(\frac{3}{2}x_1 \leq B(\mu \wedge \tilde{\tau}^+(3x_1)) \leq 2x_1, \sup_{t \leq \mu} |A(t \wedge \tilde{\tau}^+(3x_1))| \leq \frac{x_1}{2} \right) \\ &= \mathbb{P}_0 \left(\frac{3}{2}x_1 \leq B(\mu) \leq 2x_1, \sup_{t \leq \mu} |A(t \wedge \tilde{\tau}^+(3x_1))| \leq \frac{x_1}{2} \right) \\ &= \mathbb{P}_0 \left(\frac{3}{2}x_1 \leq B(\mu) \leq 2x_1 \mid \sup_{t \leq \mu} |A(t \wedge \tilde{\tau}^+(3x_1))| \leq \frac{x_1}{2} \right) \mathbb{P}_0 \left(\sup_{t \leq \mu} |A(t \wedge \tilde{\tau}^+(3x_1))| \leq \frac{x_1}{2} \right) \\ &= \mathbb{P}_0 \left(\frac{3}{2}x_1 \leq B(\mu) \leq 2x_1 \right) \mathbb{P}_0 \left(\sup_{t \leq \mu} |A(t \wedge \tilde{\tau}^+(3x_1))| \leq \frac{x_1}{2} \right) \\ &\geq C_2(\mu) \left(1 - C_1 \mu \left(\sup_{x \leq 3x_1} \sigma(x) + \sup_{x \leq 3x_1} p(x) + 1 \right) \right). \end{aligned}$$

The first equality stems from the fact that on the event

$$\left\{ \frac{3}{2}x_1 \leq B(\mu \wedge \tilde{\tau}^+(3x_1)) \leq 2x_1 \right\} \cap \left\{ \sup_{t \leq \mu} |A(t \wedge \tilde{\tau}^+(3x_1))| \leq \frac{x_1}{2} \right\},$$

μ is smaller than $\tilde{\tau}^+(3x_1)$ and the last equality is a consequence of the independence of N , Q , N_2 and B . We thus conclude that there exist $\mu_0 < \infty$ and $C(\mu_0) > 0$ such that

$$\mathbb{P}_0(\tilde{\tau}^+(x_1) \leq \mu_0) > C(\mu_0). \quad (3.19)$$

Now, let us introduce the process $Y^{(+,x_0)}$ as follows:

- $Y_0^{(+,x)} = x_0$
- $Y^{(+,x)}$ follows the SDE (3.18) on $[0, \tilde{\tau}^+(3x_1)]$
- $Y^{(+,x)}$ follows the SDE (3.16) on $[\tilde{\tau}^+(3x_1), \infty)$.

As \tilde{Y} and \tilde{Y} are non-decreasing with respect to their initial condition, by taking the same Poisson Point processes N , N_1 , N_2 and Q and the same Brownian motion B , when $Y_0 = x_0$, we can couple the processes Y and $Y^{(+,x_0)}$ to get

$$Y_t \geq Y_t^{(+,x_0)} \quad a.s. \quad \text{for any } t \geq 0.$$

Adding (3.17) we deduce that

$$\inf_{x_0 \geq 0} \mathbb{P}_{x_0}(Y_t = \infty \text{ for } t \geq \mu_0 + T_1) \geq \inf_{x_0 \geq 0} \mathbb{P}(Y_t^{(+,x_0)} = \infty \text{ for } t \geq \mu_0 + T_1) \geq C(\mu_0)\alpha.$$

Applying a renewal argument as in the proof of point (2), we get that for any $n \geq 1$,

$$\sup_{x_0 \geq 0} \mathbb{P}_{x_0}(\tau^+(\infty) \geq n(\mu_0 + T_1)) \leq (1 - C(\mu_0)\alpha)^n,$$

and thus taking the limit when n goes to infinity, we finally obtain

$$\inf_{x_0 \geq 0} \mathbb{P}_{x_0}(\tau^+(\infty) < \infty) = 1.$$

The end of the proof of point (3) to get a convergence in probability, via the Many-to-One formula, is the same as the end of the proof of point (2), and thus leads to

$$\lim_{t \rightarrow \infty} \mathbb{1}_{\{N_t \geq 1\}} \frac{\sum_{u \in V_t} \mathbb{1}_{\{\sup_{s \leq t} X_s^{(u)} < \infty\}}}{N_t} = 0, \quad \text{in probability.}$$

We now need to prove that the convergence is even almost sure. It is done exactly as in the proof of Proposition 4.1 iii) in [23] (this proof being a slight generalisation of the proof of Theorem 4.2 i) in [9]), and we refer the reader to these two papers. \square

3.3. Proof of Proposition 2.2

Let us begin with point (1). First notice that (2.5) implies that for any $x \geq 0$,

$$g(x) \vee \mathbb{E}[\mathcal{P}]r(x) \leq bx.$$

Now from Itô formula with jumps as well as from (2.5), we have the following series of inequalities, when $Y_0 = x_0$:

$$\begin{aligned} & \mathbb{E}_{x_0}[e^{b(t \wedge \tau^+(x))} Y_{t \wedge \tau^+(x)}] \\ &= x_0 + \int_0^t e^{bs} \mathbb{E}_{x_0}[\mathbb{1}_{s \leq \tau^+(x)} g(Y_s)] ds + \mathbb{E}[\mathcal{P}] \int_0^t e^{bs} \mathbb{E}_{x_0}[\mathbb{1}_{s \leq \tau^+(x)} r(\mathbb{E}_{x_0}[Y_s])] ds \\ &\leq x_0 + b \int_0^t e^{bs} \mathbb{E}_{x_0}[\mathbb{1}_{s \leq \tau^+(x)} Y_s] ds + b \int_0^t e^{bs} \mathbb{P}_{x_0}(s \leq \tau^+(x)) \mathbb{E}_{x_0}[Y_s] ds \\ &\leq x_0 + 2b \int_0^t e^{bs} \mathbb{E}_{x_0}[Y_s] ds. \end{aligned} \tag{3.20}$$

From (2.5), we have that Condition (SN ∞) is satisfied, and thus $\lim_{x \rightarrow \infty} \tau^+(x) = \infty$ a.s. Hence from Fatou's Lemma we get that

$$e^{bt} \mathbb{E}_{x_0}[Y_t] \leq x_0 + 2b \int_0^t e^{bs} \mathbb{E}_{x_0}[Y_s] ds,$$

and we conclude by Grönwall's Lemma that

$$e^{bt} \mathbb{E}_{x_0}[Y_t] \leq x_0 e^{2bt} < \infty.$$

Applying inequality (2.5) in the first line of (3.20) as well as Fatou's Lemma we obtain

$$\begin{aligned} e^{bt} \mathbb{E}_{x_0}[Y_t] &\leq x_0 + \liminf_{x \rightarrow \infty} \left(b \int_0^t e^{bs} \mathbb{P}_{x_0}(s \leq \tau^+(x)) \mathbb{E}_{x_0}[Y_s] ds \right. \\ &\quad \left. - \int_0^t \mathbb{P}_{x_0}(s \leq \tau^+(x)) (\alpha \mathbb{E}_{x_0}[Y_s] \wedge \eta) e^{bs} ds \right. \\ &\quad \left. + \int_0^t e^{bs} \mathbb{E}_{x_0}[\mathbb{1}_{s \leq \tau^+(x)} (g(Y_s) - g(\mathbb{E}_{x_0}[Y_s]))] ds \right). \end{aligned}$$

As for any $s \geq 0$, $\mathbf{1}_{s \leq \tau^+(x)}$ is smaller than one and non-decreasing with x , $g(Y_s) \leq bY_s$ and $\mathbb{E}_{x_0}[Y_s] < \infty$ we get by the Monotone Convergence Theorem,

$$e^{bt}\mathbb{E}_{x_0}[Y_t] \leq x_0 + b \int_0^t e^{bs}\mathbb{E}_{x_0}[Y_s]ds - \int_0^t (\alpha\mathbb{E}_{x_0}[Y_s] \wedge \eta)e^{bs}ds + \int_0^t e^{bs}\mathbb{E}_{x_0}[g(Y_s) - g(\mathbb{E}_{x_0}[Y_s])]ds.$$

As g is concave, Jensen inequality entails

$$e^{bt}\mathbb{E}_{x_0}[Y_t] \leq x_0 + b \int_0^t e^{bs}\mathbb{E}_{x_0}[Y_s]ds - \int_0^t (\alpha\mathbb{E}_{x_0}[Y_s] \wedge \eta)e^{bs}ds. \quad (3.21)$$

Let us introduce:

$$\mathcal{T}_1 := \inf \left\{ t \geq 0, \mathbb{E}[Y_t] < \frac{\eta}{\alpha} \right\}$$

and

$$\mathcal{T}_2 := \inf \left\{ t \geq \mathcal{T}_1, \mathbb{E}[Y_t] \geq \frac{\eta}{\alpha} \right\}$$

(if $\mathcal{T}_1 = 0$ we can choose $\eta < \alpha x_0$, which will make $\mathcal{T}_1 > 0$ and not modify the later computations). We will prove that $\mathcal{T}_1 < \infty = \mathcal{T}_2$. To this aim, let us do a reductio ad absurdum by assuming that $\mathcal{T}_1 = \infty$. In this case, (3.21) writes for any $t \geq 0$

$$e^{bt}\mathbb{E}_{x_0}[Y_t] \leq x_0 + b \int_0^t e^{bs}\mathbb{E}_{x_0}[Y_s]ds - \int_0^t \eta e^{bs}ds, \quad (3.22)$$

and if we introduce the function

$$\mathcal{G}(t) = e^{-bt} \int_0^t e^{bs}\mathbb{E}_{x_0}[Y_s]ds,$$

we obtain

$$\begin{aligned} \mathcal{G}'(t) &= -be^{-bt} \int_0^t e^{bs}\mathbb{E}_{x_0}[Y_s]ds + \mathbb{E}_{x_0}[Y_t] \\ &\leq -be^{-bt} \int_0^t e^{bs}\mathbb{E}_{x_0}[Y_s]ds + e^{-bt} \left(x_0 + b \int_0^t e^{bs}\mathbb{E}_{x_0}[Y_s]ds - \int_0^t \eta e^{bs}ds \right) \\ &= e^{-bt} \left(x_0 - \int_0^t \eta e^{bs}ds \right) \\ &= \left(x_0 + \frac{\eta}{b} \right) e^{-bt} - \frac{\eta}{b}, \end{aligned}$$

where we applied (3.22) to obtain the inequality. This implies that for every $t \geq 0$,

$$\mathcal{G}(t) \leq \left(x_0 + \frac{\eta}{b} \right) \frac{1 - e^{-bt}}{b} - \frac{\eta}{b}t$$

which is not possible as Y_t is non-negative. There is thus a contradiction and we deduce that $\mathcal{T}_1 < \infty$. Notice now that if $\mathcal{T}_2 < \infty$, then

$$e^{b\mathcal{T}_2}\mathbb{E}_{x_0}[Y_{\mathcal{T}_2}] \leq e^{b\mathcal{T}_1}\mathbb{E}_{x_0}[Y_{\mathcal{T}_1}] + (b - \alpha) \int_{\mathcal{T}_1}^{\mathcal{T}_2} e^{bs}\mathbb{E}_{x_0}[Y_s]ds,$$

which entails by an application of Grönwall's Lemma

$$\mathbb{E}_{x_0}[Y_{\mathcal{T}_2}] \leq e^{-\alpha(\mathcal{T}_2 - \mathcal{T}_1)}\mathbb{E}_{x_0}[Y_{\mathcal{T}_1}] \leq e^{-\alpha(\mathcal{T}_2 - \mathcal{T}_1)}\frac{\eta}{\alpha}.$$

This contradicts the definition of \mathcal{T}_2 , which proves that $\mathcal{T}_2 = \infty$. Notice that all the previous calculations stay true if we take a smaller η . This thus proves the following property:

$$\lim_{t \rightarrow \infty} \mathbb{E}_{x_0}[Y_t] = 0.$$

The end of the proof of point (1) follows the scheme we already used: Markov inequality to show that for any $\varepsilon > 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_{x_0}(Y_t > \varepsilon) = 0,$$

Many-to-One formula, and convergence of $N_t e^{-(b-d)t}$ on the survival event of the cell population.

Let us now prove point (2). The proof is very similar to the one of Proposition 2.1(3). Equation (3.17) holds for Y . Hence we just have to prove that for any $x_1 < \infty$ satisfying (3.17), there exists $T_0 < \infty$ such that

$$\mathbb{P}_{x_0}(\tau^+(x_1) < T_0) > 0. \quad (3.23)$$

Indeed in this case it will imply that

$$\mathbb{P}_{x_0}(\tau^+(\infty) \leq T_0 + T_1) > 0,$$

and thus

$$r(\mathbb{E}_{x_0}[Y_t]) = r(\infty) > 0, \quad \forall t \geq T_0 + T_1.$$

Indeed, as (LN_∞) implies (2.6) the expectation of Y_t stays infinity after the time $T_0 + T_1$ (see Proposition 2.3(1)). Once we have this property the end of the proof is the same.

Let us thus introduce

$$T := \tau^-(0) \wedge \tau^+(2x_1).$$

Then following the proof of Lemma 5.1 in [19] and applying Markov Inequality we get for $\delta < \infty$,

$$\mathbb{P}_{x_0} \left(\sup_{t \leq \delta} \left| \int_0^{t \wedge T} \sqrt{2\sigma^2(Y_s)} dB_s + \int_0^{t \wedge T} \int_0^{p(Y_{s-})} \int_{\mathbb{R}_+} z \tilde{Q}(ds, dx, dz) \right| > \sqrt{\delta} \right) \leq C(x_1) \sqrt{\delta}.$$

Moreover,

$$\mathbb{P}_{x_0} \left(\int_0^\delta \int_0^{2b} \int_0^1 (\theta - 1) \tilde{Y}_{s-} N(ds, dx, d\theta) = 0 \right) = e^{-2b\delta}.$$

Now if δ is small enough to satisfy $r(x_0 - \sqrt{\delta}) > r(x_0)/2 > 0$ (such a δ exists as r is a continuous function), then

$$\mathbb{P} \left(\int_0^\delta \int_0^{r(x_0)/2} \int_{\mathbb{R}_+} \mathbf{p} N_1(ds, dx, d\mathbf{p}) > 2x_1 \right) = C(x_0, x_1) > 0.$$

Combining these three computations as in the proof of Proposition 2.1(3) we get that (3.23) holds true, which ends the proof.

3.4. Proof of Proposition 2.3

Recall the definition of $G_a^{(t)}(x_0, x)$ in (3.4) and that $Z_t^{(a)}$ defined in (3.5) is a \mathcal{F}_t -martingale, and notice that under the assumptions of Proposition 2.3, as $r(\cdot) \equiv 0$, $G_a^{(t)}(x_0, x)$ does not depend on (x_0, t) . We will thus drop these dependencies in the proof. For the sake of readability we will use the following Conditions (3.24) and (3.25), equivalent to the Conditions (2.6) and (2.7), respectively:

$$\exists a \in \mathcal{A} \quad \text{and a non-negative function } f, \quad G_a(x) \geq f(x) + o(\ln x), \quad (x \rightarrow \infty), \quad (3.24)$$

$$\exists 0 < a < 1, \quad \eta > 0, \quad G_a(x) \geq \ln x (\ln \ln x)^{1+\eta}, \quad (x \rightarrow \infty). \quad (3.25)$$

This can be seen by noticing that the expressions on the left hand side of (2.6) and (2.7) equal

$$\frac{G_a(x)}{a-1} + 2b \frac{1 - \mathbb{E}[\Theta^{1-a}]}{1-a}.$$

Here again we begin with the study of the auxiliary process Y . Recall the definition of τ^- in (3.2). Then we have the two following properties:

- (a) If there exists $a \in \mathcal{A}$ and a non-negative function f such that (3.24) holds, then for any $\mathbf{b}, \mathfrak{d} > 0$,

$$\lim_{x \rightarrow \infty} \mathbb{P}_x(\tau^-(\mathbf{b}) < \mathfrak{d}) = 0. \quad (3.26)$$

(b) If there exist $0 < a < 1$, $\eta > 0$, such that (3.25) holds, then for δ small enough and \mathfrak{b} large enough,

$$\inf_{x>1} \mathbb{P}_x(\tau^-(\mathfrak{b}) \leq l(\mathfrak{b}, \delta, \eta)) \geq e^{-8\mathfrak{b}^{-\delta(1-a)}}, \quad (3.27)$$

where $l(\mathfrak{b}, \delta, \eta)$ is a function of \mathfrak{b} , δ and η , finite for \mathfrak{b} large enough and δ small enough, which will be defined in (3.28)

In words, (3.26) means that for any choice of $\mathfrak{b} < \infty$, the probability that the process reaches a value smaller than \mathfrak{b} in finite time goes to 0 when the initial condition of the process goes to infinity, and (3.27) means that for any initial condition of the process, this latter will reach a value smaller than \mathfrak{b} in finite time with a probability going to 1 when \mathfrak{b} goes to infinity. The proof of Proposition 2.3 will be a direct consequence of these two properties.

Let us begin by proving (3.26). The beginning of the proof follows ideas of the proof of [19, Theorem 2.13] but needs to be adapted as our bound is sharper and the process under consideration exhibits negative jumps unlike the processes considered in [19]. Let us take positive constants \mathfrak{d} , $\alpha < (a-1)/4\mathfrak{d}$, and \mathfrak{b} such that $G_a(x) \geq -\alpha \ln x$ holds for $x > \mathfrak{b}$. For any $0 < \varepsilon < \mathfrak{b}^{-1}$, Equation (5.17) of [19] holds. Then following the computations of [19] with $-\alpha \ln x$ in place of $-(\ln x)^r$ with $0 < r < 1$, we get for $\varepsilon^{-2^n} \leq x < \varepsilon^{-2^{n+1}}$:

$$\mathbb{P}_x \left(\tau^-(\mathfrak{b}) < \tau^+(\varepsilon^{-2^{n+1}}) \wedge \mathfrak{d} \right) \leq \mathfrak{b}^{a-1} \varepsilon^{(a-1-2\alpha\mathfrak{d})2^n} \leq \mathfrak{b}^{a-1} \varepsilon^{(a-1)2^{n-1}},$$

where in the last inequality we used that $\alpha \leq (a-1)/4\mathfrak{d}$. The end of the proof of (3.26) follows the end of the proof of point (i) of [19, Theorem 2.13].

Let us now prove (3.27). Here again we essentially follow the proof of point (ii) of [19, Theorem 2.13], but the process Y may experience negative jumps unlike the processes considered in [19], we take sharper bounds, and we need to adapt the proof. We just list the modifications to be made to cover our case. Instead of taking $t(x) = (1+\delta)^r (\ln x)^{1-r}$ (with $0 < r < 1$, $\delta > 0$), we choose

$$t(x) := \left(\ln \ln x^{1/(1+\delta)} \right)^{-(1+\eta)}, \quad \forall x > 1,$$

where η is chosen as in (2.7). Let $0 < \theta < 1$. By taking θx instead of x in the computations at the end of [19, p. 2552], we get in place of their inequality (5.18):

$$\begin{aligned} \mathbb{P}_{\theta x} \left(\tau^-(x^{(1+\delta)^{-1}}) > \tau^+(x^{(1+\delta)}) \right) &\leq \mathbb{1}_{\{\theta x > x^{(1+\delta)^{-1}}\}} \theta^{1-a} x^{-\delta(1-a)} \\ &\leq \mathbb{1}_{\{\theta x > x^{(1+\delta)^{-1}}\}} x^{-\delta(1-a)}, \end{aligned}$$

as $a < 1$. The third and fourth inequalities of [19, p. 2552] are modified as follows:

$$\begin{aligned} \theta^{1-a} x^{1-a} &\geq \mathbb{E}_{\theta x} \left[Y_{\tau^-(x^{(1+\delta)^{-1}})}^{1-a} \exp \left(\int_0^{\tau^-(x^{(1+\delta)^{-1}})} G_a(Y_s) ds \right) \mathbb{1}_{\{t(x) < \tau^-(x^{(1+\delta)^{-1}}) < \tau^+(x^{(1+\delta)})\}} \right] \\ &\geq x^{\frac{1-a}{1+\delta}} \mathbb{E}[\Theta^{1-a}] \mathbb{E}_{\theta x} \left[\exp \left(\int_0^{t(x)} \ln Y_s (\ln \ln Y_s)^{1+\eta} ds \right) \mathbb{1}_{\{t(x) < \tau^-(x^{(1+\delta)^{-1}}) < \tau^+(x^{(1+\delta)})\}} \right], \end{aligned}$$

as a negative jump may occur at time $\tau^-(x^{(1+\delta)^{-1}})$, which leads to

$$\mathbb{P}_{\theta x}(t(x) < \tau^-(x^{(1+\delta)^{-1}}) < \tau^+(x^{(1+\delta)})) \leq \mathbb{1}_{\{\theta x > x^{(1+\delta)^{-1}}\}} x^{-(1+a\delta)/(1+\delta)}.$$

If δ is small enough and x large enough, we thus still obtain the inequality:

$$\mathbb{P}_x(\tau^-(x^{(1+\delta)^{-1}})) \leq 2\mathbb{1}_{\{\theta x > x^{(1+\delta)^{-1}}\}} x^{(\delta-(1+a\delta))/(1+\delta)} \leq 2x^{-\delta(1-a)}.$$

This proves that in the computations of the beginning of [19, p. 2553], we may replace the terms of the form

$$\mathbb{P}_{\mathfrak{b}^{(1+\delta)^n}}(\cdot)$$

by terms of the form

$$\mathbb{P}_{\tau^-(\mathbf{b}(1+\delta)^n)}(\cdot)$$

without modifying the bounds. To end the proof, we still have to check that with our choice for the function t ,

$$\sum_{n=1}^{\infty} t(\mathbf{b}(1+\delta)^n) < \infty,$$

which is true. More precisely we have

$$\sum_{n=1}^{\infty} t(\mathbf{b}(1+\delta)^n) = \sum_{n=1}^{\infty} ((n-1)\ln(1+\delta) + \ln \ln \mathbf{b})^{-(1+\eta)} =: l(\mathbf{b}, \delta, \eta). \quad (3.28)$$

We may thus obtain the following inequality,

$$\inf_{x>1} \mathbb{P}_x(\tau^-(\mathbf{b}) \leq l(\mathbf{b}, \delta, \eta)) \geq e^{-8\mathbf{b}^{-\delta(1-a)}}$$

which ends the proof of (3.27).

We may now prove Proposition 2.3. Let us begin with point (1). By the Many-to-One formula (1.4), and the definition of $\tau^-(\mathbf{b})$ we have

$$e^{-(b-d)t} \mathbb{E}_x \left[\sum_{u \in V_t} \mathbb{1}_{\{X_t^u \leq \mathbf{b}\}} \right] = \mathbb{P}_x(Y_t \leq \mathbf{b}) \leq \mathbb{P}_x(\tau^-(\mathbf{b}) \leq t) \rightarrow 0, \quad (x \rightarrow \infty),$$

where we used (3.26) to obtain the last limit. We conclude as before using the convergence of $N_t e^{-(b-d)t}$ on the event of the cell population survival.

Let us now consider point (2). Let $t, \varepsilon > 0$, and δ such that (3.27) holds for \mathbf{b} large enough. By definition of $l(\mathbf{b}, \delta, \eta)$ in (3.28),

$$\lim_{\mathbf{b} \rightarrow \infty} l(\mathbf{b}, \delta, \eta) = 0.$$

We also have

$$\lim_{\mathbf{b} \rightarrow \infty} e^{-8\mathbf{b}^{-\delta(1-a)}} = 1.$$

Hence there exists $\mathbf{b}_0(t, \delta, \eta, \varepsilon)$ such that if $\mathbf{b} \geq \mathbf{b}_0(t, \delta, \eta, \varepsilon)$,

$$\inf_{x>1} \mathbb{P}_x(\tau^-(\mathbf{b}) \leq t) \geq 1 - \varepsilon.$$

Now, notice that (3.25) implies that Conditions of [19, Theorem 2.8(i)] (or its extension [21, Theorem 4.1.i]) are satisfied and thus for any $x \geq 0$,

$$\mathbb{E}_x[e^{-\tau^+(\infty)}] = 0, \quad (3.29)$$

where we recall that

$$\tau^+(\infty) := \lim_{y \rightarrow \infty} \tau^+(y).$$

As the process Y admits negative jumps, we have for any $n \in \mathbb{N}$ the inequality

$$\mathbb{E} \left[\mathbb{E}_{Y_{\tau^-(\mathbf{b})}}[e^{-\tau^+(\mathbf{b}^n)}] \right] \leq \mathbb{E}_{\mathbf{b}}[e^{-\tau^+(\mathbf{b}^n)}] + \int_0^1 \mathbb{E}_{\theta \mathbf{b}}[e^{-\tau^+(\mathbf{b}^n)}] \kappa(d\theta).$$

For any $\theta \in [0, 1]$, $\mathbb{E}_{\theta \mathbf{b}}[e^{-\tau^+(\mathbf{b}^n)}]$ is non-increasing with n , and according to (3.29)

$$\mathbb{E}_{\theta \mathbf{b}}[e^{-\tau^+(\mathbf{b}^n)}] \rightarrow 0, \quad n \rightarrow \infty.$$

By the Monotone Convergence Theorem, we thus deduce that

$$\mathbb{E} \left[\mathbb{E}_{Y_{\tau^-(\mathbf{b})}}[e^{-\tau^+(\mathbf{b}^n)}] \right] \rightarrow 0, \quad n \rightarrow \infty.$$

Hence, for any $\varepsilon, t > 0$, $\mathbf{b} > 1$, there exists $n(\mathbf{b}, t) \in \mathbb{N}$ such that for any $n \geq n(\mathbf{b})$,

$$\mathbb{E} \left[\mathbb{E}_{Y_{\tau^-(\mathbf{b})}}[e^{-\tau^+(\mathbf{b}^n)}] \right] \leq \varepsilon e^{-t},$$

which implies, by the Markov inequality,

$$\mathbb{E} \left[\mathbb{P}_{Y_{\tau^-(\mathbf{b})}}(\tau^+(\mathbf{b}^n) \leq t) \right] = \mathbb{E} \left[\mathbb{P}_{Y_{\tau^-(\mathbf{b})}}(e^{-\tau^+(\mathbf{b}^n)}) \geq e^{-t} \right] \leq \frac{\mathbb{E} \left[\mathbb{E}_{Y_{\tau^-(\mathbf{b})}}[e^{-\tau^+(\mathbf{b}^n)}] \right]}{e^{-t}} \leq \varepsilon.$$

To conclude, for any $t, \varepsilon > 0$ and δ small enough, if we choose $\mathbf{b} \geq \mathbf{b}_0(t, \delta, \eta, \varepsilon)$ and $n \geq n(\mathbf{b})$ we obtain thanks to the strong Markov inequality, and if we recall that $(\mathcal{F}_t, t \geq 0)$ is the canonical filtration associated to Y , for any $x > 1$,

$$\begin{aligned} \mathbb{P}_x(Y_t > \mathbf{b}^n) &\leq \mathbb{P}_x(\tau^-(\mathbf{b}) > t) + \mathbb{P}_x(Y_t > \mathbf{b}^n, \tau^-(\mathbf{b}) \leq t) \\ &\leq \mathbb{P}_x(\tau^-(\mathbf{b}) > t) + \mathbb{E} \left[\mathbb{P}_x(Y_t > \mathbf{b}^n, \tau^-(\mathbf{b}) \leq t | \mathcal{F}_{\tau^-(\mathbf{b})}) \right] \\ &\leq \mathbb{P}_x(\tau^-(\mathbf{b}) > t) + \mathbb{E} \left[\mathbb{P}_{Y_{\tau^-(\mathbf{b})}}(\tau^+(\mathbf{b}^n) \leq t) \right] \leq 2\varepsilon. \end{aligned}$$

We conclude as before using the Many-to-One formula (1.4) and the convergence of $N_t e^{-(b-d)t}$ on the event of the cell population survival.

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References

- [1] Gerold Alsmeyer and Sören Gröttrup. A host-parasite model for a two-type cell population. *Adv. Appl. Probab.*, 45(3):719–741, 2013.
- [2] Gerold Alsmeyer and Sören Gröttrup. Branching within branching: A model for host–parasite co-evolution. *Stochastic Processes Appl.*, 126(6):1839–1883, 2016.
- [3] Farida Nato Alzari, Shirley Longacre, Pierre Lafaye, and Jean-Claude Mazie. Phage-displayed mimotopes elicit monoclonal antibodies specific for a malaria vaccine candidate. *Biol. Chem.*, 379:65–70, 1998.
- [4] Vincent Bansaye. Proliferating parasites in dividing cells: Kimmel’s branching model revisited. *Ann. Appl. Probab.*, 18(3):967–996, 2008.
- [5] Vincent Bansaye. Cell contamination and branching processes in a random environment with immigration. *Adv. Appl. Probab.*, 41(4):1059–1081, 2009.
- [6] Vincent Bansaye, Jean-François Delmas, Laurence Marsalle, and Viet Chi Tran. Limit theorems for Markov processes indexed by continuous time Galton–Watson trees. *Ann. Appl. Probab.*, 21(6):2263–2314, 2011.
- [7] Vincent Bansaye, Juan Carlos Pardo, and Charline Smadi. On the extinction of continuous state branching processes with catastrophes. *Electron. J. Probab.*, 18:1–31, 2013.
- [8] Vincent Bansaye and Florian Simatos. On the scaling limits of Galton-Watson processes in varying environments. *Electron. J. Probab.*, 20, 2015.
- [9] Vincent Bansaye and Viet Chi Tran. Branching Feller diffusion for cell division with parasite infection. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 8:95–127, 2011.
- [10] Richard Calendar and Ross Inman. Phage biology. In *Phages: their role in bacterial pathogenesis and biotechnology*, pages 18–36. Wiley Publishing, 2005.
- [11] Bertrand Cloez. Limit theorems for some branching measure-valued processes. *Adv. Appl. Probab.*, 49(2):549–580, 2017.
- [12] Roger Detels. *Oxford textbook of global public health*, volume 2. Oxford Textbook, 2015.
- [13] Richard Durrett. Branching process models of cancer. In *Branching process models of cancer*, pages 1–63. Springer, 2015.

- [14] Hans-Otto Georgii and Ellen Baake. Supercritical multitype branching processes: the ancestral types of typical individuals. *Adv. Appl. Probab.*, 35(4):1090–1110, 2003.
- [15] Robert Hardy and Simon C. Harris. A spine approach to branching diffusions with applications to L_p -convergence of martingales. In *Séminaire de probabilités XLII*, pages 281–330. Springer, 2009.
- [16] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24. Elsevier, 1989.
- [17] Marek Kimmel. Quasistationarity in a branching model of division-within-division. In *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *The IMA Volumes in Mathematics and its Applications*, pages 157–164. Springer, 1997.
- [18] David Larocca, M. A. Brug, Kristen Jensen-Pergakes, E. Ravey, Ana Gonzalez, and Andrew Baird. Evolving phage vectors for cell targeted gene delivery. *Curr. Pharm. Biotechnol.*, 3(1):45–57, 2002.
- [19] Pei-Sen Li, Xu Yang, and Xiaowen Zhou. A general continuous-state nonlinear branching process. *Ann. Appl. Probab.*, 29(4):2523–2555, 2019.
- [20] Aline Marguet. Uniform sampling in a structured branching population. *Bernoulli*, 25:2649–2695, 2016.
- [21] Aline Marguet and Charline Smadi. Long time behaviour of continuous-state nonlinear branching processes with catastrophes. *Electron. J. Probab.*, 26:1–32, 2021.
- [22] Aline Marguet and Charline Smadi. Spread of parasites affecting death and division rates in a cell population. <https://arxiv.org/abs/2211.08265>, 2022.
- [23] Aline Marguet and Charline Smadi. Parasite infection in a cell population: role of the partitioning kernel. <https://arxiv.org/abs/2305.06962>, 2023.
- [24] Sylvie Méléard and Sylvie Roelly. Evolutive two-level population process and large population approximations. *Ann. Univ. Buchar., Math. Ser.*, 4:37–70, 2013.
- [25] Annalisa Meola, Paola Delmastro, Paolo Monaci, Alessandra Luzzago, Alfredo Nicosia, Franco Felici, Riccardo Cortese, and Giovanni Galfrè. Derivation of vaccines from mimotopes. Immunologic properties of human hepatitis B virus surface antigen mimotopes displayed on filamentous phage. *J. Immunol.*, 154(7):3162–3172, 1995.
- [26] Luis Osorio and Anita Winter. Two level branching model for virus population under cell division. <https://arxiv.org/abs/2004.14352>, 2020.
- [27] Sandra Palau and Juan Carlos Pardo. Branching processes in a Lévy random environment. *Acta Appl. Math.*, 153(1):55–79, 2018.
- [28] Philip E. Protter. Stochastic differential equations. In *Stochastic integration and differential equations*, pages 249–361. Springer, 2005.
- [29] Marjorie Russel, Henry B. Lowman, and Tim Clackson. Introduction to phage biology and phage display. In *Phage Display: A practical approach*, pages 1–26. Oxford University Press, 2004.
- [30] William O. Salivar, Helen Tzagoloff, and David Pratt. Some physical-chemical and biological properties of the rod-shaped coliphage M13. *Virology*, 24(3):359–371, 1964.
- [31] Marina Voinson, Alexandra Alvergne, Sylvain Billiard, and Charline Smadi. Stochastic dynamics of an epidemic with recurrent spillovers from an endemic reservoir. *J. Theor. Biol.*, 457:37–50, 2018.